THE UNIVERSITY OF QUEENSLAND

AUSTRALIA

Integrable boundary conditions in lattice loop models

Xavier Nev Poncini
BAdSc(Hons)[Current]

Under the supervision of
Jørgen Rasmussen

A thesis submitted to the University of Queensland
in partial fulfilment of the degree of Bachelor of Advanced Science with Honours
School of Mathematics and Physics
October 2025

Typeset in $\LaTeX\,2_\varepsilon$.

The work presented in this Thesis is, to the best of my knowledge and belief original, except as acknowledged in the text, and has not been submitted either in whole or in part, for a degree at this or any other university.

Xavier Nev Poncini

Abstract

Temperley-Lieb loop models describe systems through the connections of non-intersecting loop segments. A key feature of these models is their ability to describe non-local degrees of freedom, an illusive characteristic of physical systems. In this investigation we begin by introducing the established case of the elementary $(1,1)$ Temperley-Lieb loop model and its extension to the boundary [1]. A fusion procedure allows the construction of a $(2,2)$ Temperley-Lieb loop model from elementary $(1,1)$ objects [2]. Adapting previous methods, we introduce a new infinite class of Yang-Baxter integrable boundary conditions to the $(2,2)$ Temperley-Lieb loop model. The one-boundary $(2,2)$ Temperley-Lieb Hamiltonian is then determined. Examining the representation theory of the one-boundary $(2,2)$ Temperley-Lieb algebra $\mathcal{BTL}_N(\beta;\beta_1,\beta_2)$, permits a matrix representation of the Hamiltonian and hence a determination of the energy level structure of the model. Finally, the representation theory of the $(2,2)$ Temperley-Lieb algebra $\mathcal{TL}_N(\beta)$ is examined. We propose a closed form expression of the Gram determinant associated with the linkspace of this algebra. Preliminary investigation of this expression provides novel insights into the conditions of irreducibility of $\mathcal{TL}_N(\beta)$.

Acknowledgements

Let me begin by thanking my family, Michael, Paloma and Sharon, for your love and support in the pursuit of my passions. To my supervisor, Jørgen Rasmussen – thank you for welcoming me into your world with open arms. Through your vision, patience and generosity I have learned so much and I do not plan on leaving anytime soon. My friends, Daniel King, Declan Armstrong, Elliot Coupe, Jordan Brook, Joshua Foo, Kaiah Steven, Rebecca Haustein, Timmothy Le Pers and Tyler Jones, thank you for your warm discussions whether on physics or politics, they eased the process immensely.

Contents

List of Figures

List of Tables

List of Symbols

TL_N .... (1,1) Temperley-Lieb algebra of size N

BTL_N ...... One-boundary (1,1) Temperley-Lieb algebra of size N

$\mathcal{TL}_N$ .... (2,2) Temperley-Lieb algebra of size N

$\mathcal{BTL}_N$ ...... One-boundary (2,2) Temperley-Lieb algebra of size N

β .... Bulk loop fugacity

β_1........... Boundary loop fugacity (odd)

β_2........... Boundary loop fugacity (even)

η........... Free parameter of the model

I, e_j, e_N .... Generators of the one-boundary (1,1) Temperley-Lieb algebra

$\mathbb{I}$, $\mathbb{E}_j$, $\mathbb{E}_N$, $\mathbb{X}_j$, $\mathbb{X}_N$ Generators of the one-boundary (2,2) Temperley-Lieb algebra

$x_j(u)$, $r_j(u)$. Bulk and boundary face operators of the one-boundary (1,1) Temperley-Lieb model

$X_j(u)$, $R_j(u)$ Bulk and boundary face operators of the one-boundary (2,2) Temperley-Lieb model

$D(u)$........ Double row transfer tangle

$\mathcal{H}$........... Hamiltonian of the one-boundary (2,2) Temperley-Lieb algebra

ρ_N .... Size N representation of an element of the one-boundary (2,2) Temperley-Lieb algebra

$\overline{V}_N^k$ .... Linkspace of the (1,1) Temperley-Lieb algebra with k defects and N nodes

V_N^k.......... Linkspace of the (2,2) Temperley-Lieb algebra with k cabled defects and cabled N nodes

$\mathcal{V}_N^k$......... Linkspace of the one-boundary (2,2) Temperley-Lieb algebra with k paired boundary connections and N cabled nodes

G_N .... Gram matrix of the $(1,1)$ Temperley-Lieb algebra

$\mathcal{G}_N$ .... Gram matrix of the $(2,2)$ Temperley-Lieb algebra

1 Introduction

Integrable models offer exact solutions to the physical systems they describe, thus superseding approximate perturbative approaches. Of foundational interest is the Yang-Baxter equation (YBE), introduced independently by Yang in 1967 [3] and Baxter in 1972 [4]. This equation is synonymous with a class of local equivalence relations manifesting in a variety of different forms throughout the fields of statistical mechanics, quantum field theory, differential equations, knot theory, quantum groups, and other disciplines [5]. If a model is a solution to the Yang-Baxter equation it is integrable, facilitating a variety of approaches and allowing an exact solution to be determined [6]. The rich description provided by integrable models motivates further investigation of these systems. The utility of integrable models is demonstrated in the breadth of their application throughout aforementioned areas of research. This thesis is concerned with the former, describing statistical systems using two-dimensional (2D) lattice models.

The ubiquity of lattice models in the field of statistical mechanics arises from the natural description they provide. The physical constraints on these ensembles often produce discrete crystalline structures, the essence of which can be captured by lattice models which are manifestly discrete. The shear size and complexity of statistical systems ensure a complete characterisation is intractable. Lattice models are scalable and provide rich descriptions of the internal degrees of freedom which need not be local in nature. The success of such models was first demonstrated by Onsager [7] in 1944; producing an exact solution to the 2D Ising model [8] introduced in 1925. Followed later by the 6-vertex model in the description of ferroelectricity in 2D, Lieb [9] was the first to compute the free energy of this model in the thermodynamic limit. Later that year Sutherland, Yang and Yang [10] calculated this parameter in terms of an applied electric field.

The game was changed forever following the advent of the aforementioned Yang-Baxter equation [3, 4], ensuring the integrability of a system given a solution to this conceptually simple expression. Soon followed an explosion of 2D lattice models providing rich descriptions of the physical systems they describe. We pay particular attention to pioneering work of Nienhuis [11, 12], laying the foundation for Temperley-Lieb loop models which are the focus of this thesis. The model in question was developed by Jones [13] providing a description of planar algebras. The underlying algebraic structure was established by Temperley and Lieb [14] decades earlier. Such models have found success particularly in the description of percolation, spin-cluster and polymer-chain phenomena, of rich interest to condensed matter and material science [14–16]. Further successes have come with the introduction of integrable boundary conditions [1, 17–19], providing a complete description of systems with well defined borders, whilst also introducing additional algebraic structure.

Recent work by Morin-Duchesne, Pearce and Rasmussen concerns a fusion procedure whereby a (m, n) Temperley-Lieb loop model can be constructed from operators of the elementary $(1, 1)$ Temperley-Lieb loop model [2]. Of particular interest is the $(2, 2)$ Temperley-Lieb loop model, the structure of which has been known for some time [20, 21]. Here a spin-1 system is constructed from spin-1/2, $(1, 1)$ constituents.

This thesis extends the $(2, 2)$ Temperley-Lieb loop model allowing loop segments to terminate on the boundary. Adapting the Robin boundary conditions introduced in [1], we have produced a novel solution to the boundary Yang-Baxter equation ensuring the integrability of the one-boundary $(2, 2)$ Temperley-Lieb loop model. Applying a seam of width w to these newly established conditions, an infinite class of integrable boundary conditions are constructed. The Hamiltonian of the one-boundary $(2, 2)$ Temperley-Lieb loop model for $w = 0$ was determined as an element of the one-boundary $(2, 2)$ Temperley-Lieb algebra $\mathcal{BTL}_N(\beta; \beta_1, \beta_2)$. Examining the representation theory of $\mathcal{BTL}_N(\beta; \beta_1, \beta_2)$ we construct matrix representations of the Hamiltonian for various system sizes N. Determining the eigensystem of these representations provides insight into the energy level structure of the system. The representation theory of the $(2, 2)$ Temperley-Lieb algebra $\mathcal{TL}_N(\beta)$ is explored, we propose a closed form expression of the Gram determinant for an arbitrary system size N. Such an expression provides novel insights into the conditions of irreducibility on the linkspace associated with $\mathcal{TL}_N(\beta)$. We present a preliminary investigation examining these conditions whilst outlining future direction for a rigorous treatment.

The structure of this thesis detailing these contributions is as follows. Chapter 2 introduces the $(1, 1)$ Temperley-Lieb loop model in the bulk and on the boundary. The algebraic structure of this model is established. Chapter 3 outlines the fusion procedure allowing one to produce an (m, n) Temperley-Lieb loop model from the elementary $(1, 1)$ operators. Particular focus is placed on the $(2, 2)$ Temperley-Lieb model, where we introduce Robin boundary conditions. Here we present a general solution to the boundary Yang-Baxter equation for the $(2, 2)$ model in addition to a solution constructed from elementary $(1, 1)$ operators, ensuring the integrability of each construction. The algebraic structure of the one-boundary $(2, 2)$ Temperley-Lieb model is examined and the defining relations are listed. In Chapter 4 we construct an infinite class of boundary conditions from those introduced in Chapter 3. We show that this infinite class satisfies the boundary Yang-Baxter equation, maintaining the integrability of the model. Chapter 5 establishes the Hamiltonian of the one-boundary $(2,2)$ Temperley-Lieb loop model by taking a series expansion of the transfer tangle, a diagrammatic object generating the lattice. In Chapter 6 we explore the representation theory of the regular and one-boundary $(2,2)$ Temperley-Lieb algebras $\mathcal{TL}_N(\beta)$ and $\mathcal{BTL}_N(\beta;\beta_1,\beta_2)$ respectively. Particular focus is placed on the Hamiltonian of $\mathcal{BTL}_N(\beta;\beta_1,\beta_2)$ whose spectrum is determined, facilitating an analysis of the energy level structure of the model. The notion of an invariant bilinear form is introduced and a Gram matrix is constructed to examine the conditions of irreducibility on the linkspace of $\mathcal{TL}_N(\beta)$. A closed form expression of the Gram determinant for this linkspace is proposed. Both the zeros and poles of the Gram determinant are examined to determine crude conditions on the parameter β ensuring the irreducibility of the linkspace associated with $\mathcal{TL}_N(\beta)$. Chapter 7 provides a summary of the contributions made throughout the thesis and a lucid glance at future work, extending the progress presented in this document.

The appendices are self contained companions to the ideas presented in the thesis and can be overlooked on the first reading. Appendix A provides *Mathematica* code verifying the solutions to the boundary Yang-Baxter equation presented in Chapter 3. Appendix B lists the coefficients associated with the extended boundary conditions introduced in Chapter 4, in addition to an example calculation demonstrating this process. Appendix C presents the *Mathematica* code detailing both the Hamiltonian and Gram matrices used throughout Chapter 6, in addition to the eigenvectors of the Hamiltonians corresponding to the eigenvalues presented.

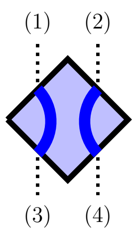
2

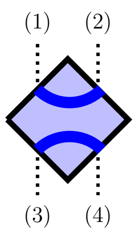
Temperley-Lieb Loop Model

In this chapter we introduce the one-boundary $(1,1)$ Temperley-Lieb loop model. The bulk and boundary face operators are presented, detailing the statistical nature of the model. The notion of integrability is established via solutions of the Yang-Baxter equations. Finally, the algebraic structure underlying the model is explored.

2.1 One-boundary $(1,1)$ Temperley-Lieb Loop Model

The $(1,1)$ Temperley-Lieb loop model is constructed from non-intersection loop segments expressed on square plaquettes. The edge of each plaquette possesses a single node, hence termed the $(1,1)$ model. Each loop segment connects two nodes on adjacent edge of the plaquette. The elementary orientations of the $(1,1)$ Temperley-Lieb loop model are given by [13]



and


. (2.1)

We interpret (1) and (2) as labelling input nodes and (3) and (4) as labelling output nodes. Neglecting labels, we observe the plaquettes are mapped to their partner via a $\pi/2$ radian rotation and to themselves via a π radian rotation. Tiling the objects of the $(1,1)$ Temperley-Lieb loop model we can construct a densely packed, non-oriented and non-intersecting connectivity map on a rectangular strip with dimension $M \times N$, where $M,\ N \in \mathbb{Z}_+$. Maps constructed solely out of square plaquettes are known as bulk configurations. See Figure

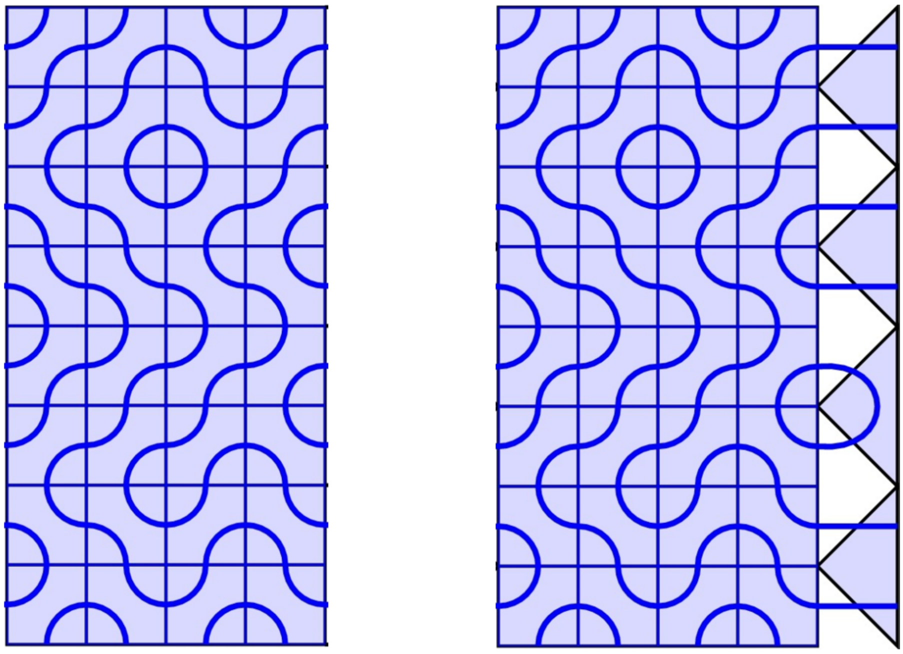
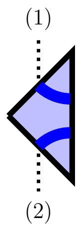


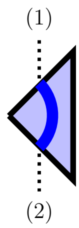
FIGURE 2.1: Implementation of the one-boundary $(1, 1)$ Temperley-Lieb elementary face operators producing a 8×4 lattice. On the left we have a bulk configuration without any boundaries. On the right, boundary triangles are attached to the far edge producing a bulk-boundary configuration.

2.1 for an example of this construction. Arising naturally in bulk configurations are closed loops, these are removed and assigned a bulk loop fugacity β.

Extending the $(1, 1)$ Temperley-Lieb loop model to the boundary we introduce boundary triangles [1]

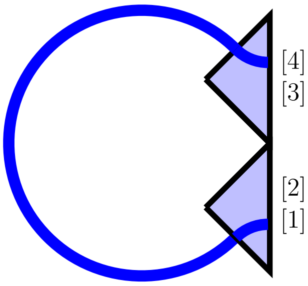


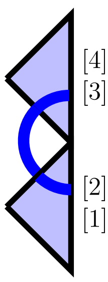
and



. (2.2)

These define the *Neumann* and *Dirichlet boundary conditions* respectively. Taking a linear combination of these diagrams we will refer to them as *Robin boundary conditions*. Maps constructed from both square and triangle plaquettes are known as bulk-boundary configurations. A lattice configuration constructed with Robin boundary conditions can be seen in Figure 2.1. Bulk-boundary configurations give rise to half loops terminating on the boundary. Boundary loops are treated in a similar manner as full loops in the bulk; they are removed and assigned a weight β_1 if the lower attachment point is odd and β_2 if this point is even. The parity of the boundary loop fugacities, β_1 and β_2 is demonstrated below

$\beta_1 \rightarrow$ 

and $\beta_2 \rightarrow$ 

.

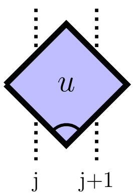
We can construct a statistical model by assigning local Boltzmann weights to each of the bulk and boundary objects presented in equations (2.1) and (2.2) respectively. These weights are interpreted as the probability of a particular plaquette existing in the lattice. Letting w_1 and w_2 label the weights associated with each of the bulk orientations expressed in equation (2.1). Similarly we assign a_1 and a_2 to each of the boundary triangles in equation (2.2). Given this construction we can compute the weight of an arbitrary lattice configuration σ

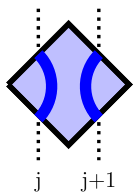
$$W_\sigma = w_1^{n_1} w_2^{n_2} a_1^{m_1} a_2^{m_2} \beta^\ell \beta_1^{\ell_1} \beta_2^{\ell_2}, \tag{2.3}$$

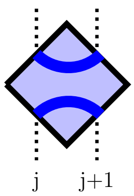
where n_1, n_2, m_1 and m_2 determine the multiplicity of each bulk and boundary objects respectively. ℓ, ℓ_1 and ℓ_2 count the number of loops existing in both the bulk and on the boundary. This formalism leads naturally to the partition function of the system. This object is simply a weighted sum over all possible lattice configurations σ

$$Z = \sum_\sigma W_\sigma. \tag{2.4}$$

Taking linear combinations of both the bulk square and boundary triangles we can construct a lattice model parameterised by u. The bulk face operator of this model is given by

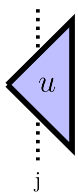
$x_j(u) =$ 

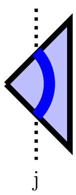
$= w_1(u)$ 

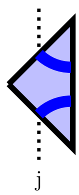
$+ w_2(u)$ 

$, \qquad u \in \mathbb{R}, \qquad (2.5)$

where $w_1(u)$ and $w_2(u)$ define the parameterisation of the local Boltzmann weights associated with each bulk orientation. The small black ark on the bottom vertex of the bulk face operator labels the orientation of the resulting bulk plaquettes. The boundary face operator is of the form

$r_j(u) =$ 

$= a_1(u)$ 

$+ a_2(u)$ 

$, \qquad (2.6)$

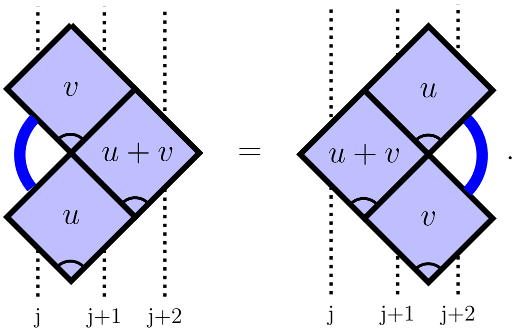
where $a_1(u)$ and $a_2(u)$ define the parameterisation of the Boltzmann weights associated with each boundary triangle configuration. Here we have introduced the one-boundary $(1,1)$ Temperley-Lieb loop model, we now seek to establish the integrability of this model.

2.2 Yang-Baxter Equations

In order for a Temperley-Lieb loop model to be Yang-Baxter integrable, the elementary face operators must satisfy both the bulk and boundary Yang-Baxter equations (YBE and BYBE). For the case of the one-boundary $(1,1)$ Temperley-Lieb loop model, we must determine the Boltzmann weights $w_1(u)$, $w_2(u)$, $a_1(u)$ and $a_2(u)$, such that each Yang-Baxter equation is satisfied. The YBE for the $(1,1)$ Temperley-Lieb loop model is of the form

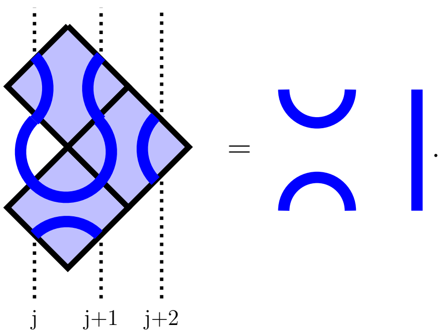
$$x_j(u)x_{j+1}(u+v)x_j(v) = x_{j+1}(v)x_j(u+v)x_{j+1}(u), \tag{2.7}$$

diagrammatically this expression is given by



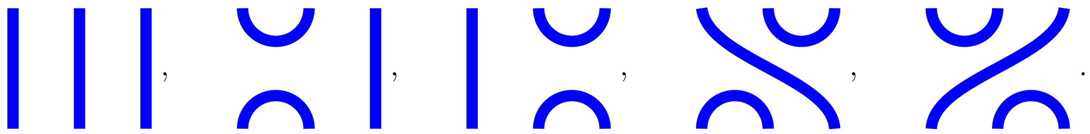
(2.8)

A solution to the YBE equation is achieved by expanding this diagrammatic object into all of the possible configurations. The bulk face operator of the $(1,1)$ Temperley-Lieb loop model possess two possible diagrammatic forms, specified in equation (2.5). Evaluating all possible diagrams constructed from this object there exists a total of 2^3 on each side of equation (2.8). Illustrating this process, let us expand one of the 16 configurations as an example



(2.9)

Performing all of the possible expansions the resulting 16 configurations fall into one of the five diagrammatic equivalence classes expressed below



Associated with each equivalence class there exists a functional equation. Assigning the Boltzmann weights and loop fugacity

$$w_1(u) = s_1(-u), \qquad w_2(u) = s_0(u), \qquad \beta = 2\cos\lambda, \tag{2.10}$$

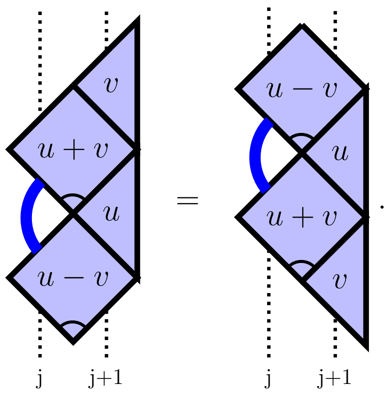
where we have adopted the succinct notation

$$s_k(u) = \frac{\sin(u + k\lambda)}{\sin\lambda}, \quad \lambda \in \mathbb{R},\ k \in \mathbb{Z}, \tag{2.11}$$

the five functional equations are simultaneously satisfied and the $(1,1)$ Temperley-Lieb loop model is integrable in the bulk. Here we have introduced the additional parameter λ, which characterises the particular $(1,1)$ Temperley-Lieb model. We direct the interested reader to the paper [15] where the full proof is presented. Let us now turn our attention toward the BYBE possessing the form

$$x_j(u-v)r_{j+1}(u)x_j(u+v)r_{j+1}(v) = r_{j+1}(v)x_j(u+v)r_{j+1}(u)x_j(u-v), \tag{2.12}$$

diagrammatically



(2.13)

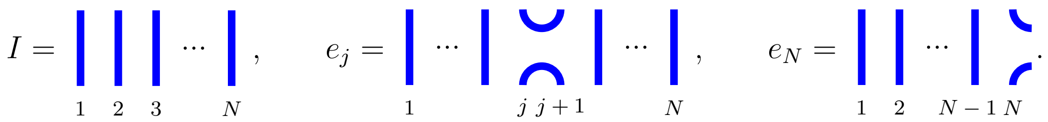
Implementing the same procedure as detailed for the YBE. A solution to the BYBE can be determined by assigning the Boltzmann weights

$$a_1(u) = \gamma - \beta_1[s_0(u)]^2 + \beta_2\left[s_0\left(u - \frac{\lambda}{2}\right)\right]^2, \qquad a_2(u) = s_0(2u), \qquad \gamma \in \mathbb{R}, \tag{2.14}$$

where γ is a free parameter and the bulk weights are those established in equation (2.10). It follow that the $(1,1)$ Temperley-Lieb loop model is integrable in the bulk and on the boundary. Again we direct the interested reader to the original paper [1] for the full proof. Here we have presented a brief introduction to the integrable one-boundary (1,1) Temperley-Lieb loop model. Let us now examine the underlying algebraic structure.

2.3 One-boundary $(1,1)$ Temperley-Lieb Algebra

Applying the elementary objects of the one-boundary $(1,1)$ Temperley-Lieb loop model we can construct the diagrammatic objects



(2.15)

Taking the vertical as the direction of transfer and assigning multiplication as the vertical concatenation of diagrams; these objects generate the one-boundary $(1,1)$ Temperley-Lieb algebra

$$BTL_N(\beta;\beta_1,\beta_2) := \langle I, e_j; j = 1, ..., N\rangle. \tag{2.16}$$

This algebra is an extension of the $(1,1)$ Temperley-Lieb algebra

$$TL_N(\beta) := \langle I, e_j; j = 1, ..., N-1\rangle, \tag{2.17}$$

where the operator e_N is introduced. The algebras $BTL_N(\beta;\beta_1,\beta_2)$ and $TL_N(\beta)$ share the defining relations [14]

$$[e_i,\ e_j] = 0, \quad |i-j| > 1 \tag{2.18}$$

$$e_i e_j e_i = e_i, \quad |i-j| = 1 \tag{2.19}$$

$$e_j^2 = \beta e_j, \quad j = 1, ..., N-1 \tag{2.20}$$

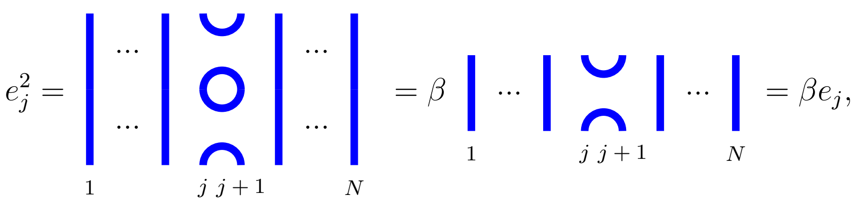
with those specific to $BTL_N(\beta;\beta_1,\beta_2)$ given by

$$[e_j, e_N] = 0, \quad j = 1, ..., N-2 \tag{2.21}$$

$$e_{N-1}e_N e_{N-1} = \beta_1 e_{N-1}, \tag{2.22}$$

$$e_N^2 = \beta_2 e_N. \tag{2.23}$$

Let us demonstrate the notion of multiplication by showing equation (2.20) holds



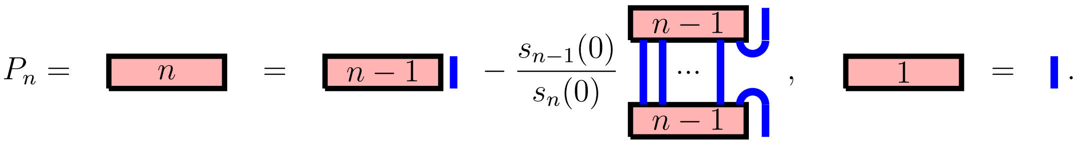
as required.

It should be noted that the one-boundary $(1,1)$ Temperley-Lieb algebra $BTL_N(\beta;\beta_1,\beta_2)$ is a generalisation of the blob algebra $\mathcal{B}_N(\beta,\beta')$ [22]. These algebras are isomorphic given $\beta_2 \neq 0$

$$BTL_N(\beta;\beta_1,\beta_2) \simeq \mathcal{B}_N(\beta,\beta'), \qquad \text{where } \beta' = \frac{\beta_1}{\beta_2}. \tag{2.24}$$

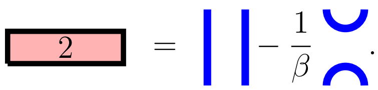
2.4 Wenzl-Jones Projector

Motivated by the fusion procedure which will be developed in Chapter 3, we must introduce a key ingredient, the Wenzl-Jones (WJ) projector [23–25]. This object is defined recursively as



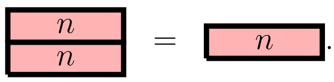
(2.25)

Examining the first non-trivial case $n = 2$, we have



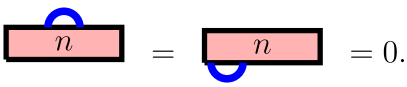
(2.26)

This object is a projector in the sense that, $P_n^2 = P_n$. Diagrammatically we have



(2.27)

The projector is annihilated if there exists any half loops resting on the edge



(2.28)

As will be clear in the following section, these properties are essential in the construction of fused face operators.

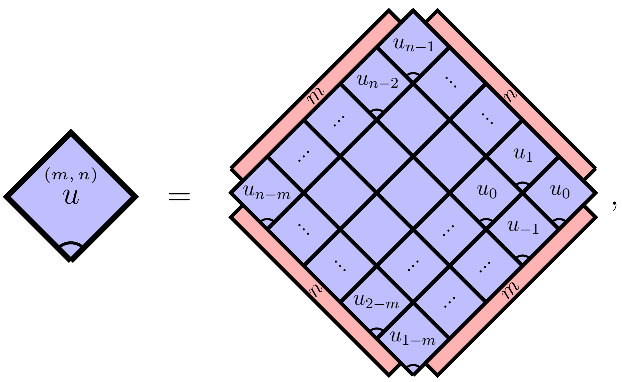
3

Fused Temperley-Lieb Loop Models

We are now in a position to introduce fused Temperley-Lieb loop models. Such models are constructed from the elementary $(1,1)$ bulk operators defined in Chapter 2, ultimately allowing one to produce a (m,n) Temperley-Lieb lattice loop model. In this investigation we examine the $(2,2)$ Temperley-Lieb lattice loop model where we introduce a new class of boundary conditions. A novel solution to the boundary Yang-Baxter equation is presented ensuring the integrability of the model. This general solution is then reconstructed by detailing a fusion procedure in terms of the elementary $(1,1)$ objects. The chapter is concluded by exploring the underlying algebraic structure of the model.

3.1 (m,n) Temperley-Lieb Loop Model

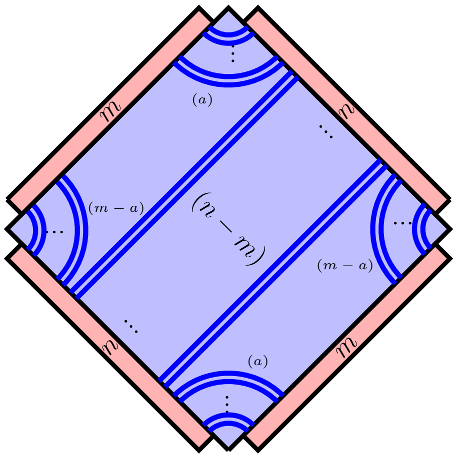
Employing the fusion procedure discussed in [2], one can construct bulk face operators of arbitrary size. This is achieved by forming a rectangular array of m by n elementary face operators and placing P_m and P_n projectors on each of the respective edges. The resulting (m,n) fused face operator is expressed diagrammatically as



where $u_m = u + m\lambda$. (3.1)

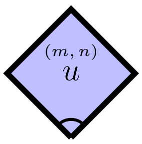
The expansion of the (m, n) fused face operator yields $2^{m\times n}$ possible diagrams. Exploiting the properties of the WJ projectors on the edges of the face operator, one can dramatically simplify this object in terms of internal connections [2]

$$X_a^{m,n} =$$



$$, \qquad n \geq m. \tag{3.2}$$

Re-expressing the (m, n) operator from equation (3.1) in terms of this object, we have [2]

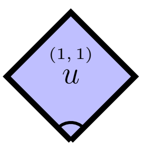


$$= \sum_{a=0}^{r} \alpha_a^{m,n} X_a^{m,n}, \qquad r := \min(m, n), \tag{3.3}$$

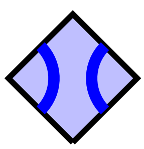
where

$$\alpha_a^{m,n} = (-1)^{a(m+n)} \left(\prod_{i=0}^{m-1} \prod_{j=0}^{n-1} s_{i-j+1}(-u) \right) \left(\prod_{k=1}^{a} \frac{s_{r-k+1}(0)}{s_k(0)} \right) \left(\prod_{l=0}^{a-1} \frac{s_{n-r+l}(u)}{s_{m-l}(-u)} \right). \tag{3.4}$$

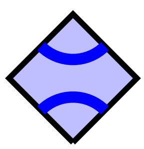
It follows that the Boltzmann weights associated with the general (m, n) face operator presented in equation (3.3), readily satisfies the (m, n) YBE. See [2] for a full treatment. Specialising this result to $m = n = 1$, we recover the bulk face operator of the $(1, 1)$ Temperley-Lieb loop model



$$= s_1(-u)X_0^{1,1} + s_0(u)X_1^{1,1}, \quad \text{where} \quad X_0^{1,1} =$$

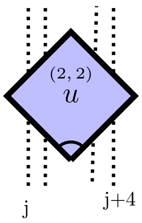


$$, \qquad X_1^{1,1} =$$

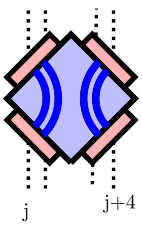


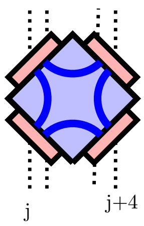
$$. \tag{3.5}$$

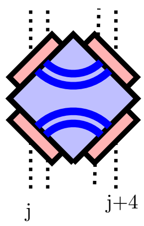
Setting $m = n = 2$ we have the bulk face operator of the $(2, 2)$ Temperley-Lieb loop model

$X_j(u) =$ 

$= s_{-1}(u)s_0(u)$ (3.6)

$\Bigg(s_{-2}(u)s_{-1}(u)$ 

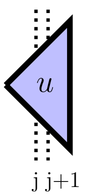
$- \beta s_{-1}(u)s_0(u)$ 

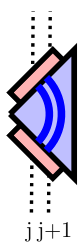
$+ s_1(u)s_0(u)$ 

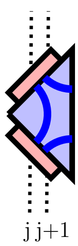
$\Bigg).$ (3.7)

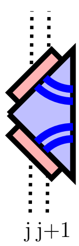
Four elementary $(1,1)$ plaquettes were applied in the construction of each $(2,2)$ object. Examining the weights associated with the $(2,2)$ objects we have a product of four functions. Each function is the contribution of the weight associated with the $(1,1)$ plaquettes used in the construction of the $(2,2)$ object. For the remainder of this document we will be dealing with $(2,2)$ operators, for convenience we have dropped the 2 label from the projectors.

Let us now introduce a $(2,2)$ boundary face operator such that the $(2,2)$ Temperley-Lieb loop model can be extended to the boundary. This construction seeks to capture all possible configurations connecting two input and output strands whilst allowing connections to and from the boundary. As for all diagrammatic Temperley-Lieb objects, the crossing of loop segments is forbidden. The general $(2,2)$ boundary face operator is of the form

$R_j(u) =$ 

$= A(u)$ 

$+ B(u)$ 

$+ C(u)$ 

. (3.8)

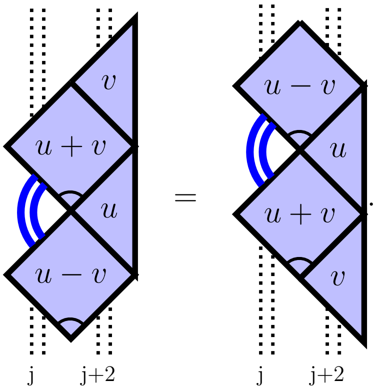
Collectively we will refer to this construction as the *Robin boundary conditions*. Applying the $(2,2)$ bulk and boundary operators to the BYBE, we seek to determine the *general* form of the functions $A(u)$, $B(u)$ and $C(u)$ such that the BYBE is satisfied. Establishing the integrability of the model.

3.2 Boundary Yang-Baxter Equation

The algebraic form of the $(2, 2)$ BYBE is given by

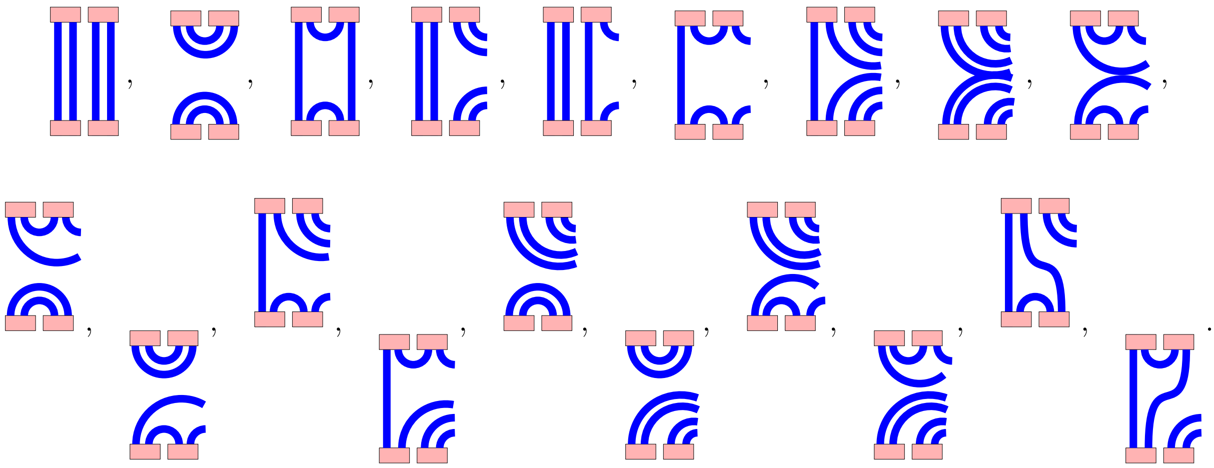
$$X_j(u-v)R_{j+2}(u)X_j(u+v)R_{j+2}(v) = R_{j+2}(v)X_j(u+v)R_{j+2}(u)X_j(u-v), \tag{3.9}$$

diagrammatically

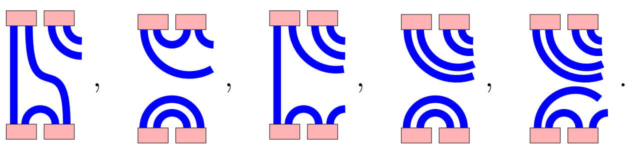


(3.10)

Applying the $(2, 2)$ bulk and boundary face operators to the BYBE, this object decomposes into 3^4 connectivity diagrams on each side of equation (3.10). Each of the 162 diagrams are then grouped into 19 inequivalent classes, presented below



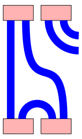
Symmetry about the horizontal ensure the coefficients of the first nine diagrams readily match for all u, v. Each of the non-horizontal symmetric diagrams possess a horizontal flip pair. The resulting functional equations produced by equating coefficients are identical. It follows that we need only to focus on the five diagrams



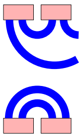
Equating the coefficients of these diagrams we have five functional equations that must be simultaneously satisfied. After considerable simplification and applying the definitions

$$\mathcal{X}(v) := \frac{A(v)}{C(v)}, \qquad \mathcal{Y}(v) := \frac{B(v)}{C(v)}, \tag{3.11}$$

we have the five functional equations.

I. 

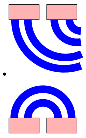
$$\begin{aligned}
\beta\beta_1\Big\{\sin\lambda\sin(2v)\Big\}\mathcal{X}(v) \\
-\beta\beta_1\Big\{\sin\lambda\sin(2u)\Big\}\mathcal{X}(u) \\
+\beta_1\sin(u-v)\Big\{\beta_2\sin(u+v-2\lambda)-\beta_1\sin(u+v-\lambda)\Big\}\mathcal{Y}(v) \\
+\beta_1\sin(u-v)\Big\{\beta_2\sin(u+v-2\lambda)-\beta_1\sin(u+v-\lambda)\Big\}\mathcal{Y}(u) \\
+\beta\Big\{\sin\lambda\sin(2v)\Big\}\mathcal{X}(v)\mathcal{Y}(u) \\
-\beta\Big\{\sin\lambda\sin(2u)\Big\}\mathcal{Y}(v)\mathcal{X}(u) \\
+\sin(u-v)\Big\{\beta_2\sin(u+v-2\lambda)-\beta_1\sin(u+v-\lambda)\Big\}\mathcal{Y}(v)\mathcal{Y}(u) \\
+\beta_1^2\sin(u-v)\Big\{\beta_2\sin(u+v-2\lambda)-\beta_1\sin(u+v-\lambda)\Big\} = 0
\end{aligned}$$

II. 

$$
\begin{aligned}
&-\beta\left(\beta_2-\frac{2\beta_1}{\beta}\right)\sin\lambda\Big\{\sin(u+v)\sin(u-v)\sin(2v)\Big\}\mathcal{X}(v)\\
&-\sin(u+v)\sin(u-v)\Big[\beta_2\sin(u+v-\lambda)-\beta_1\sin(u+v)\Big]\\
&\qquad\Big[\beta_2\sin(u-v+\lambda)-\beta_1\sin(u-v)\Big]\mathcal{Y}(v)\\
&-\sin^2\lambda\Big\{\sin(2v)\sin(2u+\lambda)\Big\}\mathcal{X}(v)\mathcal{Y}(u)\\
&+\sin^2\lambda\Big\{\sin(2u)\sin(2u+\lambda)\Big\}\mathcal{Y}(v)\mathcal{X}(u)\\
&-\sin\lambda\sin(u-v)\sin(2u+\lambda)\Big\{\beta_2\sin(u+v-\lambda)-\beta_1\sin(u+v)\Big\}\mathcal{Y}(v)\mathcal{Y}(u)=0
\end{aligned}
$$

III. 

$$
\begin{aligned}
&-\beta\sin\lambda\Big\{\sin(2v)\Big\}\mathcal{X}(v)\\
&+\beta\sin\lambda\Big\{\sin(2u)\Big\}\mathcal{X}(u)\\
&+\Big\{-\beta\sin\lambda\left(\beta_2-\frac{\beta_1}{\beta}\right)\sin(2v)+\left(\frac{2\beta_1}{\beta}-\beta_2\right)\sin(u-v)\Big\}\mathcal{Y}(v)\\
&+\Big\{\beta\sin\lambda\left(\beta_2-\frac{\beta_1}{\beta}\right)\sin(2u)+\left(\frac{2\beta_1}{\beta}-\beta_2\right)\sin(u-v)\Big\}\mathcal{Y}(u)\\
&-\frac{1}{\beta}\sin\lambda\Big\{\sin(2u+\lambda)-\sin(2v+\lambda)\Big\}\mathcal{Y}(v)\mathcal{Y}(u)\\
&+\Big\{\beta\left(\beta_2-\frac{\beta_1}{\beta}\right)\sin(u-v)\Big[\beta_2\sin(u+v)-\beta_1\sin(u+v-\lambda)\Big]\\
&\qquad+\beta_1\left(\frac{2\beta_1}{\beta}-\beta_2\right)\sin(u+v)\sin(u-v)\Big\}=0
\end{aligned}
$$

IV. 

$$
\begin{aligned}
\beta\sin\lambda\sin(2v)\Big\{\cos(2\lambda)\cos(2v)-\cos\lambda\cos(2u-\lambda)\Big\}\mathcal{X}(v)\\
+\beta\sin^2\lambda\Big\{\sin(2u)\sin(2u+\lambda)\Big\}\mathcal{X}(u)\\
+\sin(u+v-\lambda)\sin(u-v)\\
\Big\{(\beta\beta_2-\beta_1)\sin(u-v+\lambda)\sin(u+v-\lambda)-\beta_1\sin(u+v)\sin(u-v)\Big\}\mathcal{Y}(v)\\
+\sin\lambda\sin(u-v)\sin(2u+\lambda)\Big\{-(\beta_2\beta-2\beta_1)\sin(u+v-\lambda)+\beta_1\sin(u+v+\lambda)\Big\}\mathcal{Y}(u)\\
+\sin\lambda\Big\{\sin(u+v-\lambda)\sin(u-v)\sin(2u+\lambda)\Big\}\mathcal{Y}(v)\mathcal{Y}(u)\\
+\Big\{-\beta\beta_2^2\sin(u+v-\lambda)\sin(u+v)\sin(u-v)\sin(u-v+\lambda)\\
+\beta_1\beta_2\beta\sin(u-v)\Big[\sin(u+v-\lambda)[\cos\lambda\cos(2v-\lambda)-\cos(2u)]+\sin^2(u+v)\sin(u-v+\lambda)\Big]\\
-\beta_1^2\sin(u-v)\Big[\sin^2(u+v-\lambda)\sin(u-v+\lambda)\\
+\sin(u+v)\sin(u-v)[2\sin(u+v-\lambda)+\sin(u+v+\lambda)]\Big]\Big\}=0
\end{aligned}
$$

V. 

$$
\begin{aligned}
\beta\sin\lambda\sin(2v+\lambda)\mathcal{Y}(v)\\
-\beta\sin\lambda\sin(2u+\lambda)\mathcal{Y}(u)\\
+\beta\,(2\beta_1-\beta\beta_2)\sin(u-v)\sin(u+v)=0.
\end{aligned}
$$

3.2.1 Solution

The previous section established simplified forms of the five functional equations that must be simultaneously satisfied to produce a solution to the BYBE. Let us now detail the process applied to determine a solution. Further manipulations of functional equation I allows us to write this relation as

$$
\begin{aligned}
&\left(\beta\sin\lambda\sin(2v)\Big[\beta_1+\mathcal{Y}(v)\Big]^{-1}\mathcal{X}(v)-\sin(v)\Big(\beta_2\sin(v-2\lambda)-\beta_1\sin(v-\lambda)\Big)\right) \\
-&\left(\beta\sin\lambda\sin(2u)\Big[\beta_1+\mathcal{Y}(u)\Big]^{-1}\mathcal{X}(u)-\sin(u)\Big(\beta_2\sin(u-2\lambda)-\beta_1\sin(u-\lambda)\Big)\right)=0.
\end{aligned}
\tag{3.12}
$$

The general solution of this expression is given by

$$
\mathcal{X}_\eta(v)=\frac{1}{\beta\sin\lambda\sin(2v)}\Big[\beta_1+\mathcal{Y}(v)\Big]\Big[\eta+\sin(v)(\beta_2\sin(v-2\lambda)-\beta_1\sin(v-\lambda))\Big],\quad \eta\in\mathbb{R},
\tag{3.13}
$$

where η is a free parameter of the model. Applying similar techniques to functional equation V, we have

$$
\Big(\sin\lambda\sin(2v+\lambda)\mathcal{Y}(v)-(2\beta_1-\beta\beta_2)\sin^2(v)\Big)-\Big(\sin\lambda\sin(2u+\lambda)\mathcal{Y}(u)-(2\beta_1-\beta\beta_2)\sin^2(u)\Big)=0.
\tag{3.14}
$$

The general solutions of this expression is given by

$$
\mathcal{Y}_\gamma(v)=\frac{1}{\sin\lambda\sin(2v+\lambda)}\Big[\gamma+(2\beta_1-\beta\beta_2)\sin^2(v)\Big],\quad \gamma\in\mathbb{R},
\tag{3.15}
$$

where γ is another free parameter of the model. Applying the functions $\mathcal{X}_\eta(v)$ and $\mathcal{Y}_\gamma(v)$, to the functional equation III, this expression can be simplified

$$
\begin{aligned}
&\left\{\gamma\Big(\frac{1}{\sin\lambda\sin(2u+\lambda)}-\frac{1}{\sin\lambda\sin(2v+\lambda)}\Big)+(2\beta_1-\beta\beta_2)\Big(\frac{\sin^2(u)}{\sin\lambda\sin(2u+\lambda)}-\frac{\sin^2(v)}{\sin\lambda\sin(2v+\lambda)}\Big)\right\} \\
&\Big[\Big\{\eta\beta-(\beta\beta_2-\beta_1)\sin^2\lambda\Big\}+\gamma\Big]=0.
\end{aligned}
\tag{3.16}
$$

Equation (3.16) must hold for all u, v this is true if we set

$$
\gamma=(\beta\beta_2-\beta_1)\sin^2\lambda-\eta\beta.
\tag{3.17}
$$

It follows we have $\mathcal{X}_\eta(v)$ and $\mathcal{Y}_\eta(v)$

$$
\mathcal{X}_\eta(v):=\frac{A(v)}{C(v)}=\frac{1}{\beta\sin\lambda\sin(2v)}\Big[\beta_1+\mathcal{Y}_\eta(v)\Big]\Big[\eta+\sin(v)(\beta_2\sin(v-2\lambda)-\beta_1\sin(v-\lambda))\Big],
\tag{3.18}
$$

$$\mathcal{Y}_\eta(v) := \frac{B(v)}{C(v)} = \frac{1}{\sin\lambda\sin(2v+\lambda)}\Big[(\beta\beta_2-\beta_1)\sin^2\lambda - \eta\beta + (2\beta_1-\beta\beta_2)\sin^2(v)\Big]. \quad (3.19)$$

Employing the assistance of *Mathematica*, functional equations II and IV are satisfied after applying the forms of $\mathcal{X}_\eta(v)$ and $\mathcal{Y}_\eta(v)$. See appendix A.1.1 for the full *Mathematica* code verifying this solution. As the solutions are given as ratios of our desired functions, we have the freedom to define an overall normalisation coefficient. This normalisation is assigned to ensure, $\lim_{v\to 0} A(v) = 1$ and $\lim_{v\to 0} B(v) = \lim_{v\to 0} C(v) = 0$. The resulting expressions are given by

$$\begin{aligned} A_\eta(v) =& \left[\frac{\sin\lambda}{\eta(\eta-\beta_2\sin^2\lambda)\sin(2v+\lambda)}\right]\Big[\eta - \beta_1\sin(v+\lambda)\sin(v) + \beta_2\sin(v+\lambda)\sin(v-\lambda)\Big] \\ &\Big[\eta - \beta_1\sin(v)\sin(v-\lambda) + \beta_2\sin(v)\sin(v-2\lambda)\Big], \end{aligned} \quad (3.20)$$

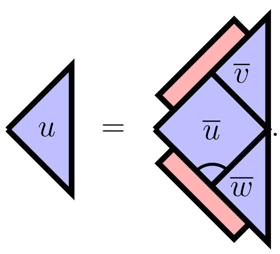
$$B_\eta(v) = \frac{\sin^2\lambda\sin(2v)}{\eta(\beta_2\sin^2\lambda-\eta)\sin(2v+\lambda)}\Big[(\beta\beta_2-\beta_1)\sin^2\lambda - \eta\beta + (2\beta_1-\beta\beta_2)\sin^2(v)\Big], \quad (3.21)$$

$$C_\eta(v) = \frac{\sin^3\lambda\sin(2v)}{\eta(\beta_2\sin^2\lambda-\eta)}. \quad (3.22)$$

Assigning these functions to equation (3.8), ensures the integrability of the one-boundary $(2,2)$ Temperley-Lieb loop model. Again, we encourage the reader to verify the forms of $A_\eta(v)$, $B_\eta(v)$ and $C_\eta(v)$ provide a solution to the BYBE by implementing the full *Mathematica* code in appendix A.1.2.

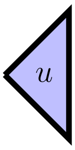
3.3 Fused Boundary Operator

The previous section detailed a general solution to the BYBE, establishing the integrability of the Robin boundary conditions presented in equation (3.8). We now seek to determine a fusion procedure such that the Robin boundary conditions can be constructed from elementary $(1,1)$ bulk and boundary operators. Let us begin with two $(1,1)$ boundary face operators possessing a common vertical edge and a single $(1,1)$ bulk face operator resting in the centre of these objects. On each of the diagonal edges we place a WJ projector. Diagrammatically this object is of the form

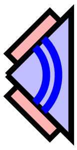


(3.23)

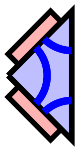
The arguments of the $(1,1)$ face operators remain arbitrary and are assigned the labels $\overline{u}$, $\overline{v}$ and $\overline{w}$. Expanding this object we have



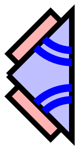
$$= w_1(\overline{u})a_1(\overline{v})a_1(\overline{w})$$



$$+ w_1(\overline{u})[a_1(\overline{v})a_2(\overline{w}) + a_2(\overline{v})a_1(\overline{w}) + \beta_2 a_2(\overline{v})a_2(\overline{w})] \tag{3.24}$$



$$+ w_2(\overline{u})a_2(\overline{v})a_2(\overline{w})$$



.

Here we have recovered the three boundary triangles detailed in equation (3.8). Let us now examine the corresponding coefficients, assigning the definitions

$$\mathcal{A}(u) := w_1(\overline{u})a_1(\overline{v})a_1(\overline{w}) \tag{3.25}$$
$$\mathcal{B}(u) := w_1(\overline{u})[a_1(\overline{v})a_2(\overline{w}) + a_2(\overline{v})a_1(\overline{w}) + \beta_2 a_2(\overline{v})a_2(\overline{w})] \tag{3.26}$$
$$\mathcal{C}(u) := w_2(\overline{u})a_2(\overline{v})a_2(\overline{w}). \tag{3.27}$$

We seek to determine $\overline{u}$, $\overline{v}$ and $\overline{w}$ such that $A_\eta(u) = \mathcal{A}(u)$, $B_\eta(u) = \mathcal{B}(u)$ and $C_\eta(u) = \mathcal{C}(u)$ up to an arbitrary normalisation factor. Applying the known forms of $w_1(x)$, $w_2(x)$, $a_1(x)$ and $a_2(x)$, we have

$$\mathcal{A}(u) = \frac{\sin(\lambda-\overline{u})}{\sin\lambda}\left(\gamma - \beta_1\left[\frac{\sin(\overline{v})}{\sin\lambda}\right]^2 + \beta_2\left[\frac{\sin\left(\overline{v}-\frac{\lambda}{2}\right)}{\sin\lambda}\right]^2\right)\left(\gamma - \beta_1\left[\frac{\sin(\overline{w})}{\sin\lambda}\right]^2 + \beta_2\left[\frac{\sin\left(\overline{w}-\frac{\lambda}{2}\right)}{\sin\lambda}\right]^2\right), \tag{3.28}$$

$$\begin{aligned}\mathcal{B}(u) = \frac{\sin(\lambda-\overline{u})}{\sin\lambda}\Bigg[&\frac{\sin(2\overline{w})}{\sin\lambda}\left(\gamma - \beta_1\left[\frac{\sin(\overline{v})}{\sin\lambda}\right]^2 + \beta_2\left[\frac{\sin\left(\overline{v}-\frac{\lambda}{2}\right)}{\sin\lambda}\right]^2\right)\\ &+ \frac{\sin(2\overline{v})}{\sin\lambda}\left(\gamma - \beta_1\left[\frac{\sin(\overline{w})}{\sin\lambda}\right]^2 + \beta_2\left[\frac{\sin\left(\overline{w}-\frac{\lambda}{2}\right)}{\sin\lambda}\right]^2\right) + \beta_2\frac{\sin(2\overline{v})}{\sin\lambda}\frac{\sin(2\overline{w})}{\sin\lambda}\Bigg],\end{aligned} \tag{3.29}$$

$$\mathcal{C}(u) = \frac{\sin(\overline{u})\sin(2\overline{v})\sin(2\overline{w})}{\sin^3\lambda}. \tag{3.30}$$

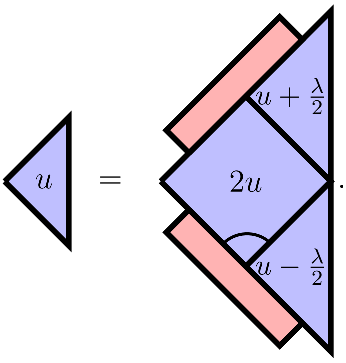
Employing various trigonometric identities, we notice a striking similarity to the desired forms of these expressions by setting $\overline{u} = 2u$, $\overline{v} = u + \frac{\lambda}{2}$ and $\overline{w} = u - \frac{\lambda}{2}$. It follows that

$$\mathcal{A}(u) = -\frac{\sin(2u-\lambda)}{\sin^5\lambda}\Big(\eta - \beta_1 \sin(u+\lambda)\sin(u) + \beta_2 \sin(u-\lambda)\sin(u+\lambda)\Big)$$
$$\Big(\eta - \beta_1 \sin(u)\sin(u-\lambda) + \beta_2 \sin(u-2\lambda)\sin(u)\Big), \tag{3.31}$$

$$\mathcal{B}(u) = \frac{\sin(2u-\lambda)\sin(2u)}{\sin^4\lambda}\Big((\beta\beta_2 - \beta_1)\sin^2\lambda - \eta\beta + (2\beta_1 - \beta\beta_2)\sin^2(u)\Big), \tag{3.32}$$

$$\mathcal{C}(u) = \frac{\sin(2u)\sin(2u+\lambda)\sin(2u-\lambda)}{\sin^3\lambda}.$$

Comparing these coefficients with those satisfying the BYBE presented in equations (3.20) – (3.22), we see they are the same up to a constant factor. It follows that the general solution to the BYBE can be recovered by decomposing the boundary operator expressed diagrammatically as

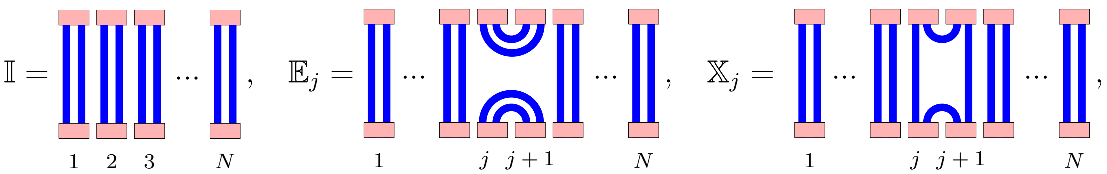


(3.33)

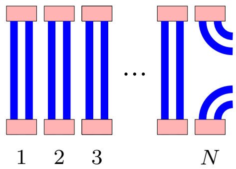
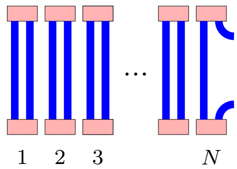
Here we have detailed a fusion procedure for the boundary operator of the one-boundary $(2,2)$ Temperley-Lieb loop model. This result lays the foundation to construct a (n,n) boundary operator. Pairing the (n,n) boundary operator with the (n,n) bulk operator defined in equation (3.1), we envisage the construction of an integrable one-boundary (n,n) Temperley-Lieb loop model.

3.4 One-boundary $(2,2)$ Temperley-Lieb Algebra

Applying the operators of the one-boundary $(2,2)$ Temperley-Lieb loop model we can construct the diagrammatic objects



(3.34)

$\mathbb{E}_N =$  , $\quad \mathbb{X}_N =$  . (3.35)

Taking the vertical as the direction of transfer and defining multiplication as the vertical concatenation of diagrams. These objects generate the one-boundary $(2,2)$ Temperley-Lieb algebra

$$\mathcal{BTL}_N(\beta;\beta_1,\beta_2) := \langle \mathbb{I}, \mathbb{E}_j, \mathbb{X}_j; j = 1, ..., N\rangle. \tag{3.36}$$

This algebra is an extension of the $(2,2)$ Temperley-Lieb algebra

$$\mathcal{TL}_N(\beta) := \langle \mathbb{I}, \mathbb{E}_j, \mathbb{X}_j; j = 1, ..., N-1\rangle, \tag{3.37}$$

where the operators $\mathbb{E}_N$ and $\mathbb{X}_N$ are introduced. The algebras $\mathcal{BTL}_N(\beta;\beta_1,\beta_2)$ and $\mathcal{TL}_N(\beta)$ share the defining relations [26],

$$\begin{aligned}
\mathbb{E}_j &= \left(\beta^2-1\right)\mathbb{E}_j && (3.38)\\
\mathbb{E}_j\mathbb{E}_{j\pm1}\mathbb{E}_j &= \mathbb{E}_j && (3.39)\\
\beta^2\mathbb{X}_j^2 &= \beta\left(\beta^2-2\right)\mathbb{X}_j + \mathbb{E}_j && (3.40)\\
\beta\mathbb{X}_j\mathbb{E}_j &= \beta\mathbb{E}_j\mathbb{X}_j = \left(\beta^2-1\right)\mathbb{E}_j && (3.41)\\
\mathbb{E}_i\mathbb{E}_j &= \mathbb{E}_j\mathbb{E}_i, \quad |i-j|\geq 2 && (3.42)\\
\mathbb{X}_i\mathbb{X}_j &= \mathbb{X}_j\mathbb{X}_i, \quad |i-j|\geq 2 && (3.43)\\
\mathbb{X}_i\mathbb{E}_j &= \mathbb{E}_j\mathbb{X}_i, \quad |i-j|\geq 2 && (3.44)\\
\mathbb{X}_j\mathbb{E}_{j\pm1}\mathbb{X}_j &= \mathbb{X}_{j\pm1}\mathbb{E}_j\mathbb{X}_{j\pm1} && (3.45)\\
\mathbb{E}_j\mathbb{X}_{j\pm}\mathbb{E}_j &= \left(\beta-\frac{1}{\beta}\right)\mathbb{E}_j && (3.46)\\
\mathbb{X}_j\mathbb{E}_{j\pm1}\mathbb{E}_j &= \mathbb{X}_{j\pm1}\mathbb{E}_j && (3.47)\\
\mathbb{X}_j\mathbb{X}_{j\pm1}\mathbb{E}_j &= \left(\beta-\frac{2}{\beta}\right)\mathbb{X}_{j\pm1}\mathbb{E}_j + \frac{1}{\beta^2}\mathbb{E}_j && (3.48)
\end{aligned}$$

$$\beta^3\left(\mathbb{X}_j\mathbb{X}_{j+1}\mathbb{X}_j - \mathbb{X}_{j+1}\mathbb{X}_j\mathbb{X}_{j+1}\right) = \beta\left(\mathbb{E}_j\mathbb{X}_{j+1} - \mathbb{E}_{j+1}\mathbb{X}_j + \mathbb{X}_{j+1}\mathbb{E}_j - \mathbb{X}_j\mathbb{E}_{j+1} + \mathbb{X}_j - \mathbb{X}_{j+1}\right) - \mathbb{E}_j + \mathbb{E}_{j+1}. \tag{3.49}$$

With the boundary defining relations specific to $\mathcal{BTL}_N(\beta;\beta_1,\beta_2)$ listed below

$$\mathbb{E}_N^2 = \beta_1 \left(\beta_2 - \frac{\beta_1}{\beta}\right) \mathbb{E}_N \tag{3.50}$$

$$\mathbb{X}_N^2 = \beta_2 \mathbb{X}_N - \frac{1}{\beta}\mathbb{E}_N \tag{3.51}$$

$$\mathbb{E}_N \mathbb{X}_N = \left(\beta_2 - \frac{\beta_1}{\beta}\right) \mathbb{E}_N \tag{3.52}$$

$$\mathbb{X}_{N-1}\mathbb{E}_N\mathbb{E}_{N-1} = \left(\beta_2 - \frac{\beta_1}{\beta}\right) \mathbb{X}_N\mathbb{E}_{N-1} + \frac{\beta_1^2}{\beta^2}\mathbb{E}_{N-1} \tag{3.53}$$

$$\mathbb{E}_{N-1}\mathbb{E}_N\mathbb{X}_{N-1} = \left(\beta_2 - \frac{\beta_1}{\beta}\right) \mathbb{E}_{N-1}\mathbb{X}_N + \frac{\beta_1^2}{\beta^2}\mathbb{E}_{N-1} \tag{3.54}$$

$$\mathbb{X}_{N-1}\mathbb{X}_N\mathbb{E}_{N-1} = \left(\beta - \frac{2}{\beta}\right) \mathbb{X}_N\mathbb{E}_{N-1} + \frac{\beta_1}{\beta^2}\mathbb{E}_{N-1} \tag{3.55}$$

$$\mathbb{E}_{N-1}\mathbb{X}_N\mathbb{X}_{N-1} = \left(\beta - \frac{2}{\beta}\right) \mathbb{E}_{N-1}\mathbb{X}_N + \frac{\beta_1}{\beta^2}\mathbb{E}_{N-1} \tag{3.56}$$

$$+\mathbb{X}_N\mathbb{X}_{N-1}\mathbb{X}_N - \frac{\beta_1}{\beta}\mathbb{X}_N\mathbb{E}_{N-1}\mathbb{X}_N + \frac{1}{\beta}\Big[\mathbb{E}_N\mathbb{X}_{N-1} + \mathbb{X}_{N-1}\mathbb{E}_N - \beta_1\Big(\mathbb{X}_N\mathbb{X}_{N-1} + \mathbb{X}_{N-1}\mathbb{X}_N\Big)\Big]$$

$$\mathbb{E}_{N-1}\mathbb{E}_N\mathbb{E}_{N-1} = \beta_1 \left(\beta_2 - \frac{\beta_1}{\beta}\right) \mathbb{E}_N \tag{3.57}$$

$$\mathbb{E}_{N-1}\mathbb{X}_N\mathbb{E}_{N-1} = \beta_1 \left(\beta - \frac{1}{\beta}\right) \mathbb{E}_{N-1} \tag{3.58}$$

$$\mathbb{X}_{N-1}\mathbb{E}_N\mathbb{X}_{N-1} = \frac{\beta_1}{\beta^2}\Big[\mathbb{X}_N\mathbb{E}_{N-1} + \mathbb{E}_{N-1}\mathbb{X}_N + \beta_1\Big(\mathbb{X}_{N-1} - \frac{1}{\beta}\mathbb{E}_{N-1}\Big)\Big] - \frac{1}{\beta^2}\mathbb{E}_N \tag{3.59}$$

$$\mathbb{X}_{N-1}\mathbb{X}_N\mathbb{X}_{N-1} = \frac{1}{\beta^2}(\mathbb{X}_N\mathbb{E}_{N-1} + \mathbb{E}_{N-1}\mathbb{X}_N) + \frac{\beta_1}{\beta^2}(\mathbb{X}_{N-1} - \frac{1}{\beta}\mathbb{E}_{N-1}) \tag{3.60}$$

$$+\left(\frac{\beta - \frac{3}{\beta}}{\beta_2 - \frac{2\beta_1}{\beta}}\right)\left[\mathbb{X}_N\mathbb{X}_{N-1}\mathbb{X}_N - \frac{1}{\beta}\Big[\beta_1(\mathbb{X}_N\mathbb{X}_{N-1} + \mathbb{X}_{N-1}\mathbb{X}_N) - (\mathbb{E}_N\mathbb{X}_{N-1} + \mathbb{X}_{N-1}\mathbb{E}_N)\Big] - \frac{1}{\beta^2}\mathbb{E}_N\right]$$

$$\mathbb{X}_N\mathbb{X}_{N-1}\mathbb{E}_N = -\mathbb{E}_N\mathbb{X}_{N-1}\mathbb{X}_N + 2\beta_1\left(\frac{\beta_2 - \frac{\beta_1}{\beta}}{\beta_2 - \frac{2\beta_1}{\beta}}\right)\mathbb{X}_N\mathbb{X}_{N-1}\mathbb{X}_N \tag{3.61}$$

$$+\frac{\beta_1}{\beta^2}\left[1 + 2\left(\frac{\beta_2 - \frac{\beta_1}{\beta}}{\beta_2 - \frac{2\beta_1}{\beta}}\right)\right]\mathbb{E}_N - \left(\frac{2}{\beta(\beta_2 - \frac{2\beta_1}{\beta})}\right)\left[\mathbb{E}_N\mathbb{X}_{N-1}\mathbb{E}_N - \frac{\beta_1^2}{\beta^2}\mathbb{E}_{N-1}\right]$$

$$+\left(\beta_2 - \frac{\beta_1}{\beta}\right)\left[1 + \frac{2\beta_1}{\beta(\beta_2 - \frac{2\beta_1}{\beta})}\right]\left[\mathbb{E}_N\mathbb{X}_{N-1} + \mathbb{X}_{N-1}\mathbb{E}_N - \beta_1(\mathbb{X}_N\mathbb{X}_{N-1} + \mathbb{X}_{N-1}\mathbb{X}_N)\right] \tag{3.62}$$

$$[\mathbb{E}_N, \mathbb{E}_{N-n}] = 0, \quad n \geq 2 \tag{3.63}$$
$$[\mathbb{E}_N, \mathbb{X}_{N-n}] = 0, \quad n \geq 2 \tag{3.64}$$
$$[\mathbb{X}_N, \mathbb{E}_{N-n}] = 0, \quad n \geq 2 \tag{3.65}$$
$$[\mathbb{X}_N, \mathbb{X}_{N-n}] = 0, \quad n \geq 2 \tag{3.66}$$
$$[\mathbb{E}_N, \mathbb{X}_N] = 0. \tag{3.67}$$

Here we have defined the one-boundary $(2,2)$ Temperley-Lieb algebra $\mathcal{BTL}_N(\beta;\beta_1,\beta_2)$. We now seek to extend the Robin boundary conditions introduced in this chapter.

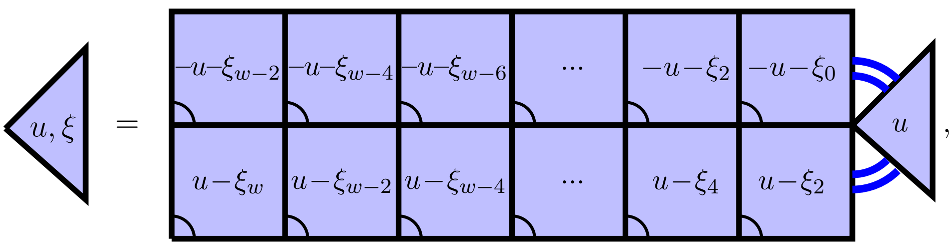
4

Extended Boundary Conditions

The previous chapter introduced new integrable boundary conditions to the $(2,2)$ Temperley-Lieb loop model permitting three distinct behaviours at the boundary. Extending the width of the Robin boundary triangle we can construct an infinite class of integrable boundary conditions permitted by the one-boundary $(2,2)$ Temperley-Lieb loop model.

4.1 Extended Robin Boundary Conditions

Applying a boundary seam of width $w \in 2\mathbb{N}_0$ to the central vertex of the $(2,2)$ boundary triangle operator, the extended boundary conditions are given by

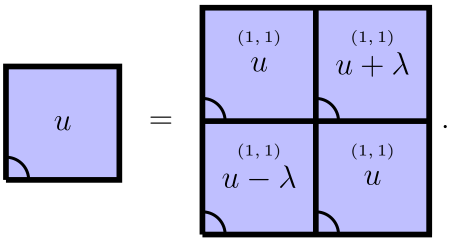


, (4.1)

where

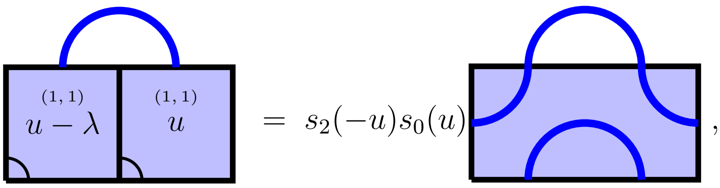
$$\xi_k = \xi + k\lambda.$$

Here ξ is a parameter of the model. Specialising to the case $w = 0$ we recover the simple Robin boundary triangle. The boundary seam is constructed from bulk square operators of the $(2,2)$ Temperley-Lieb loop model. Expressing them in terms of $(1,1)$ objects we have



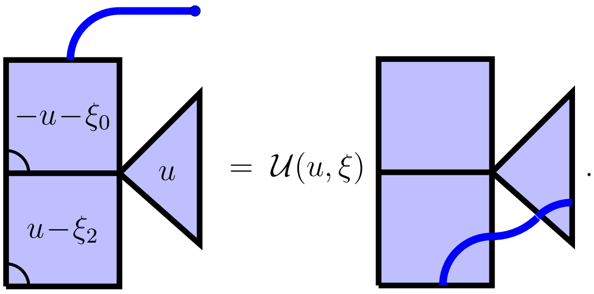
(4.2)

Substituting this form of the $(2,2)$ operator into equation (4.1) we see the difference between the argument of each $(1,1)$ operator is λ. Exploiting the drop-down property of $(1,1)$ operators



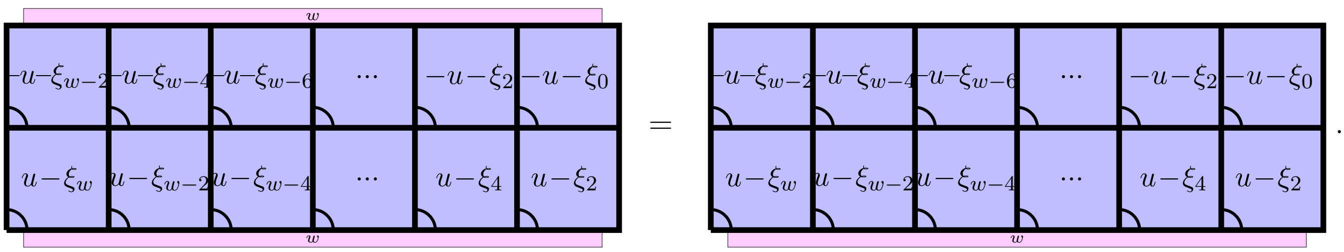
(4.3)

we see that any half loop existing on the top edge ensures a companion loop is also present on the bottom edge. We would like to develop the drop-down property for loops terminating on the boundary, this condition is as follows



(4.4)

Assigning a particular value to the free parameter η, we seek to eliminate contributions from other connectivity diagrams ensuring the one expressed above is the only form remaining. Such a construction is yet to be completed and will be a future direction of the project. The action of the drop-down properties for half loops terminating on the edge and those on the boundary ensure it suffices for there to exist one projector resting on the bottom edge, diagrammatically

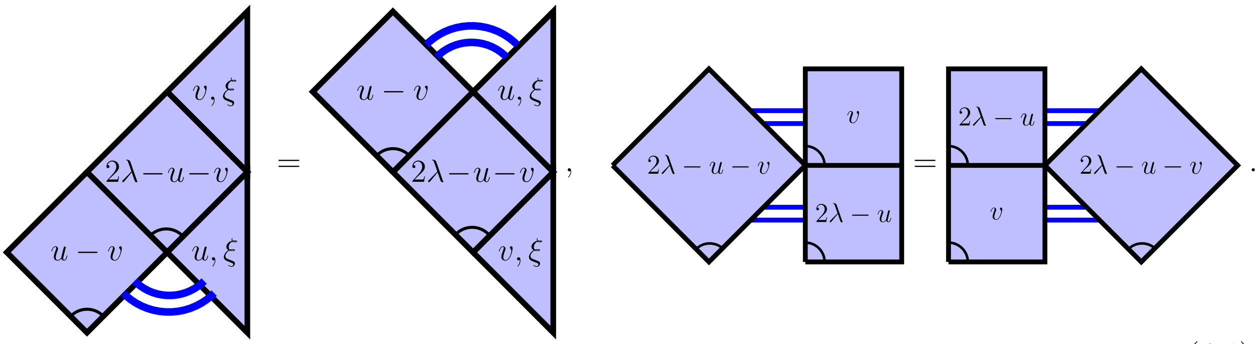


(4.5)

Here we have introduced a generalised form of the WJ projector. This object eliminates both half loops and boundary connections resting on its surface. The rigorous mathematical construction of the boundary-Wenzl Jones projector is yet to be developed and is an immediate future direction of the project. For the purpose of this investigation this object will be enforced by hand.

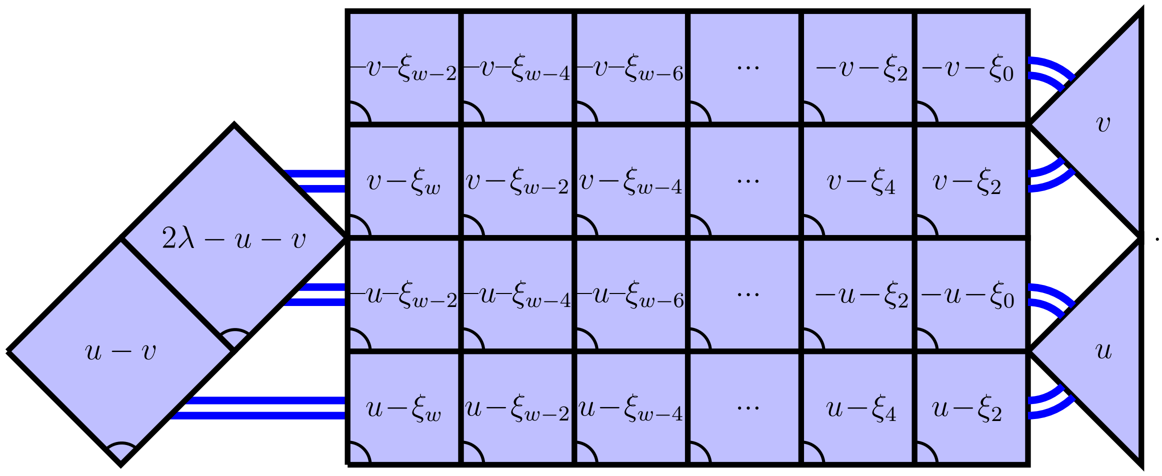
4.2 Boundary Yang Baxter Equation

We seek to maintain the integrability of the Robin boundary conditions given the application of the boundary seam. It follows we must show the BYBE is satisfied. Alternative yet equivalent forms of both the BYBE and the YBE used throughout this section and are given by [1]

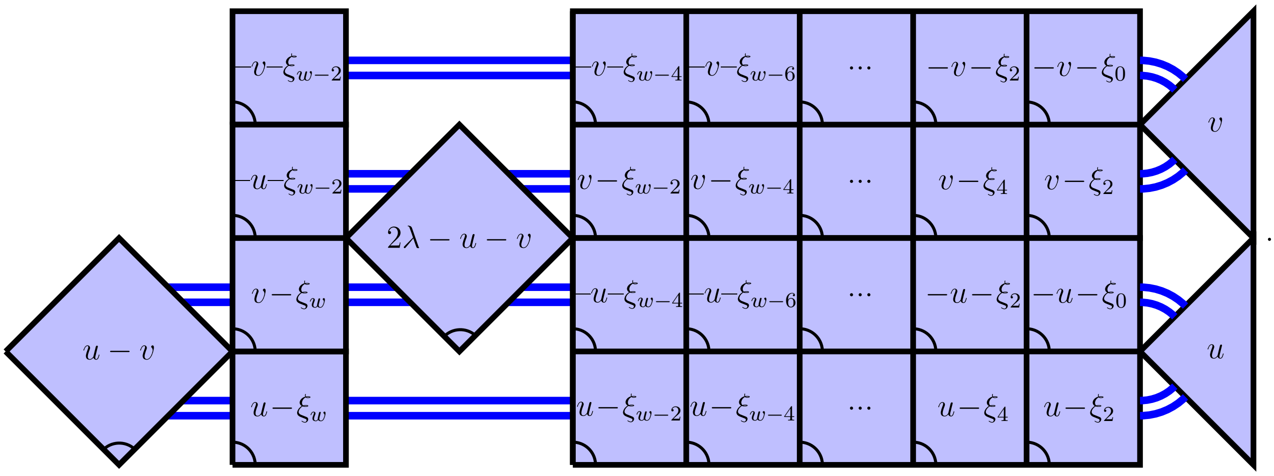


(4.6)

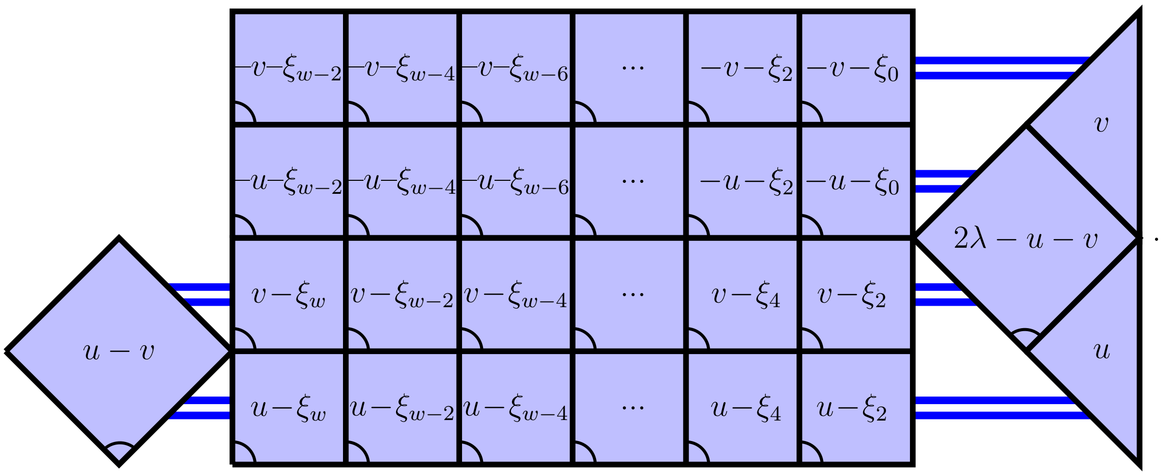
Let us first expand the left hand side (LHS) of the BYBE

$LHS =$ 

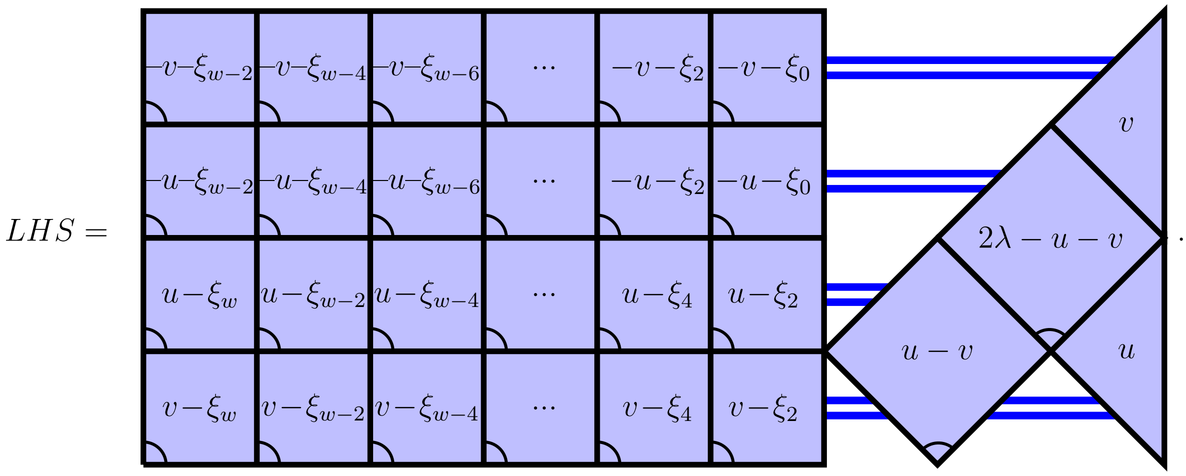
Applying the YBE yields



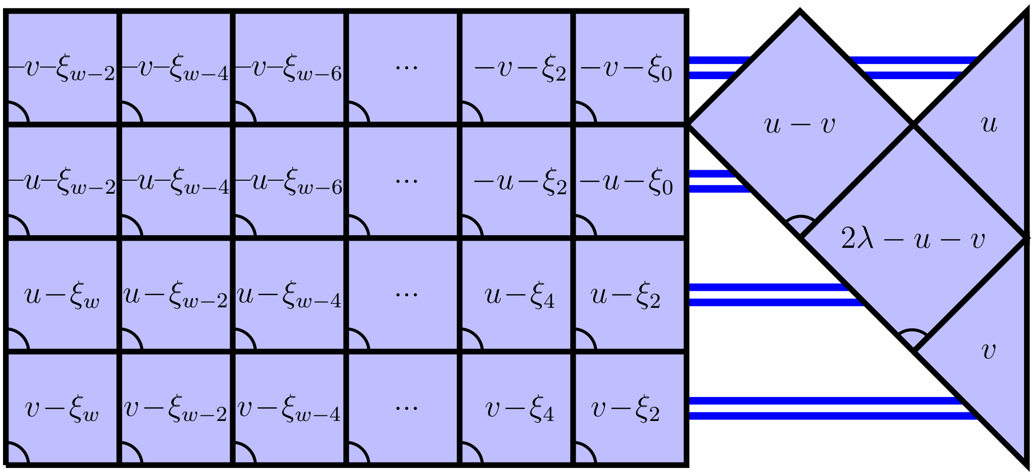
Iterating this process we can move this large square to rest between the boundary triangles. In doing so we swap the labels on the two central blocks



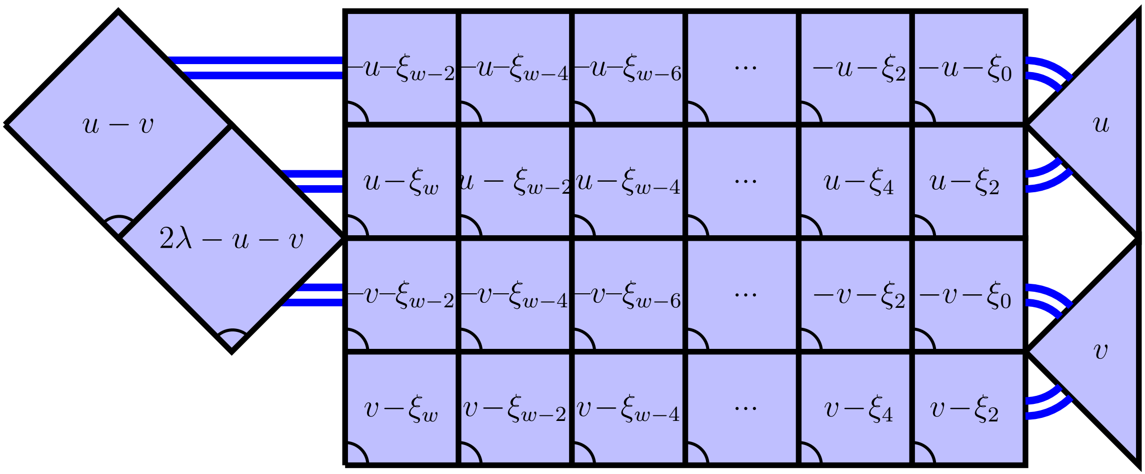
Again we iteratively apply the YBE such that the remaining large square is moved to the far right, consequently we change the labels on the lower two blocks

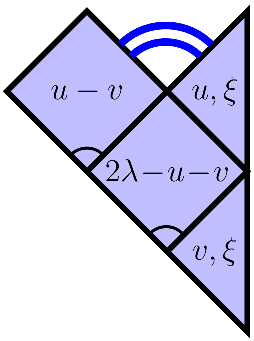


Now applying the BYBE equation (4.6), to flip the object on the right hand side

$LHS =$ 

In a similar way that the squares were translated from left to right they can be translated from right to left through iterative application of the YBE, producing the result

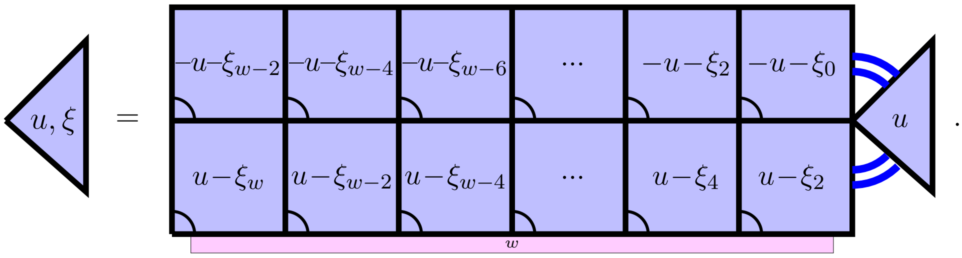
$LHS =$ 

$LHS =$  $= RHS.$

Arriving at the right hand side, it follows that the BYBE expressed in equation (4.6) is satisfied. Here we have introduced an infinite class of integrable boundary conditions to the $(2, 2)$ Temperley-Lieb model.

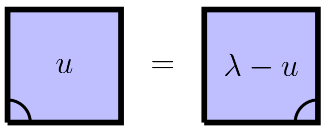
4.3 Decomposition

From the drop-down property detailed in the previous section we apply a boundary-Wenzl Jones projector to the bottom edge of the boundary seam. The boundary-WJ projector forbids diagrams possessing half loops on the bottom edge and connections from the edge to the boundary. This is enforced by the properties of the projector prescribed in Section 4.1. Applying this object we have



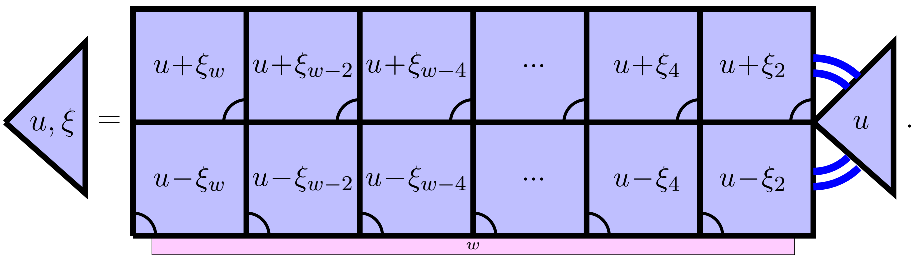
. (4.7)

Exploiting crossing symmetry



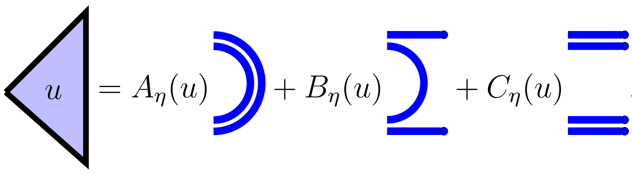
,

at the level of both the $(1,1)$ and $(2,2)$ operators, the boundary seam is equivalently expressed



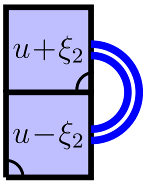
. (4.8)

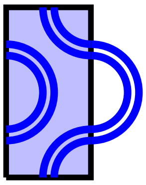
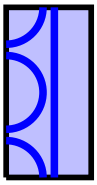
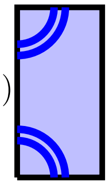
To examine how this object decomposes let us examine the system for small boundary seams and then generalise this result. For $w = 0$ we have

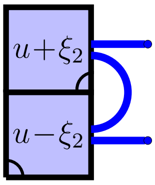


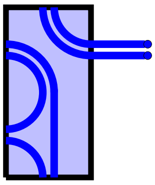
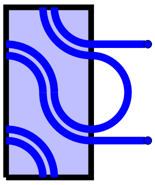
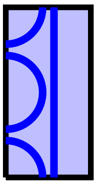
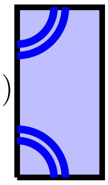
, (4.9)

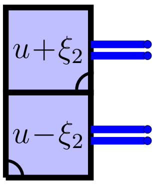
where the coefficients $A_\eta(u)$, $B_\eta(u)$ and $C_\eta(u)$, are defined in equations (3.20) – (3.22). In the interest of the reader the coefficients of the following diagrams are presented in appendix B.1. Applying the previous forms to the smallest boundary seam $w = 2$, we have

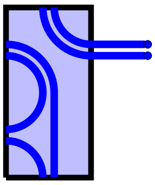
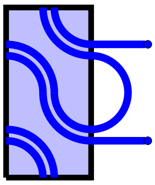
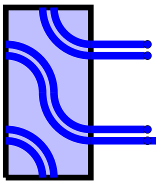


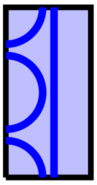
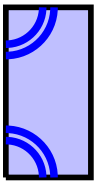
$= a^{(2)}(u,\xi_2)$  $+\, b^{(2)}(u,\xi_2)$  $+\, c^{(2)}(u,\xi_2)$  (4.10)



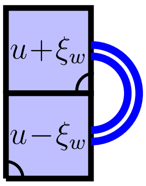
$= r^{(2)}(u,\xi_2)$  $+\, s^{(2)}(u,\xi_2)$  $+\, t^{(2)}(u,\xi_2)$  $+\, u^{(2)}(u,\xi_2)$  (4.11)

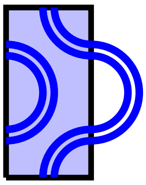
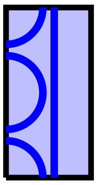
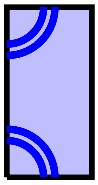


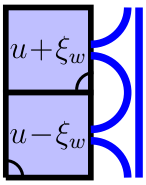
$= v^{(2)}(u,\xi_2)$  $+\, w^{(2)}(u,\xi_2)$  $+\, x^{(2)}(u,\xi_2)$  (4.12)

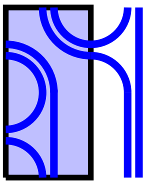
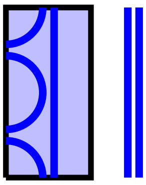
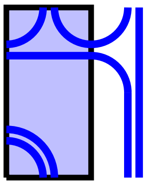
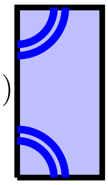
$+y^{(2)}(u,\xi_2)$  $+\, z^{(2)}(u,\xi_2)$  . (4.13)

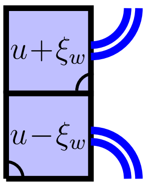
Here we recognise seven forms that reoccur for arbitrary seam widths.

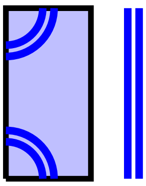
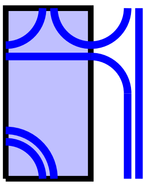
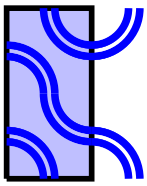


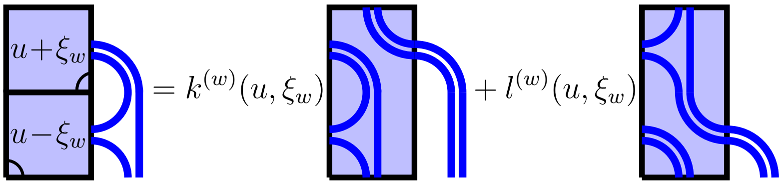
$= a^{(w)}(u,\xi_w)$  $+\, b^{(w)}(u,\xi_w)$  $+\, c^{(w)}(u,\xi_w)$  (4.14)



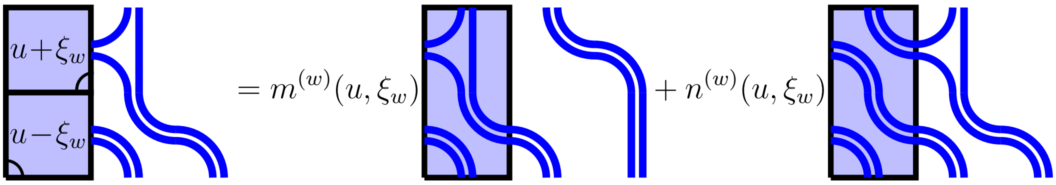
$= d^{(w)}(u,\xi_w)$  $+\, e^{(w)}(u,\xi_w)$  $+\, f^{(w)}(u,\xi_w)$  $+\, g^{(w)}(u,\xi_w)$  (4.15)



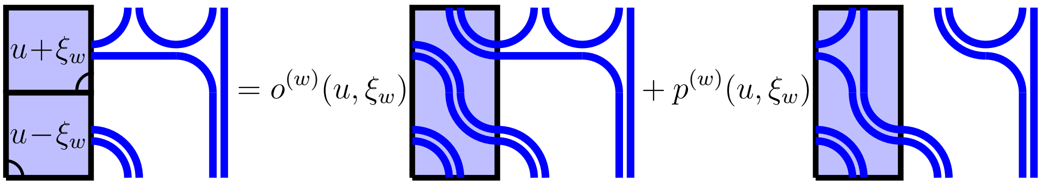
$= h^{(w)}(u,\xi_w)$  $+\, i^{(w)}(u,\xi_w)$  $+\, j^{(w)}(u,\xi_w)$  (4.16)



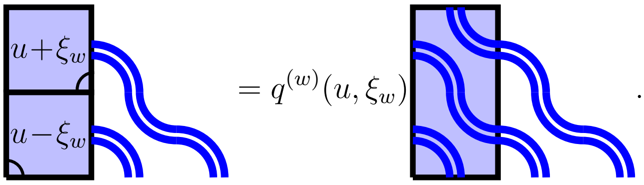
(4.17)



(4.18)

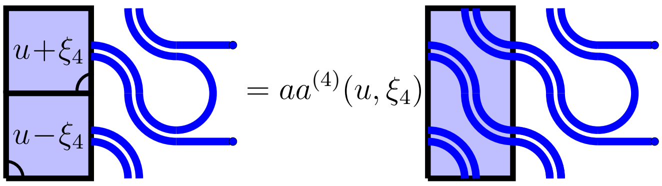


(4.19)

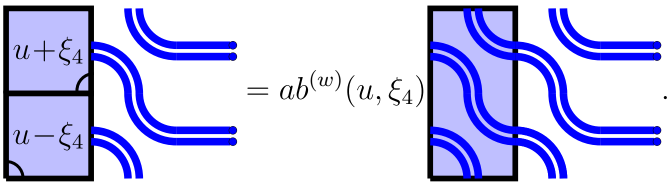


(4.20)

We also observe two forms that only occur close to the boundary, for $w = 4$ these are given by

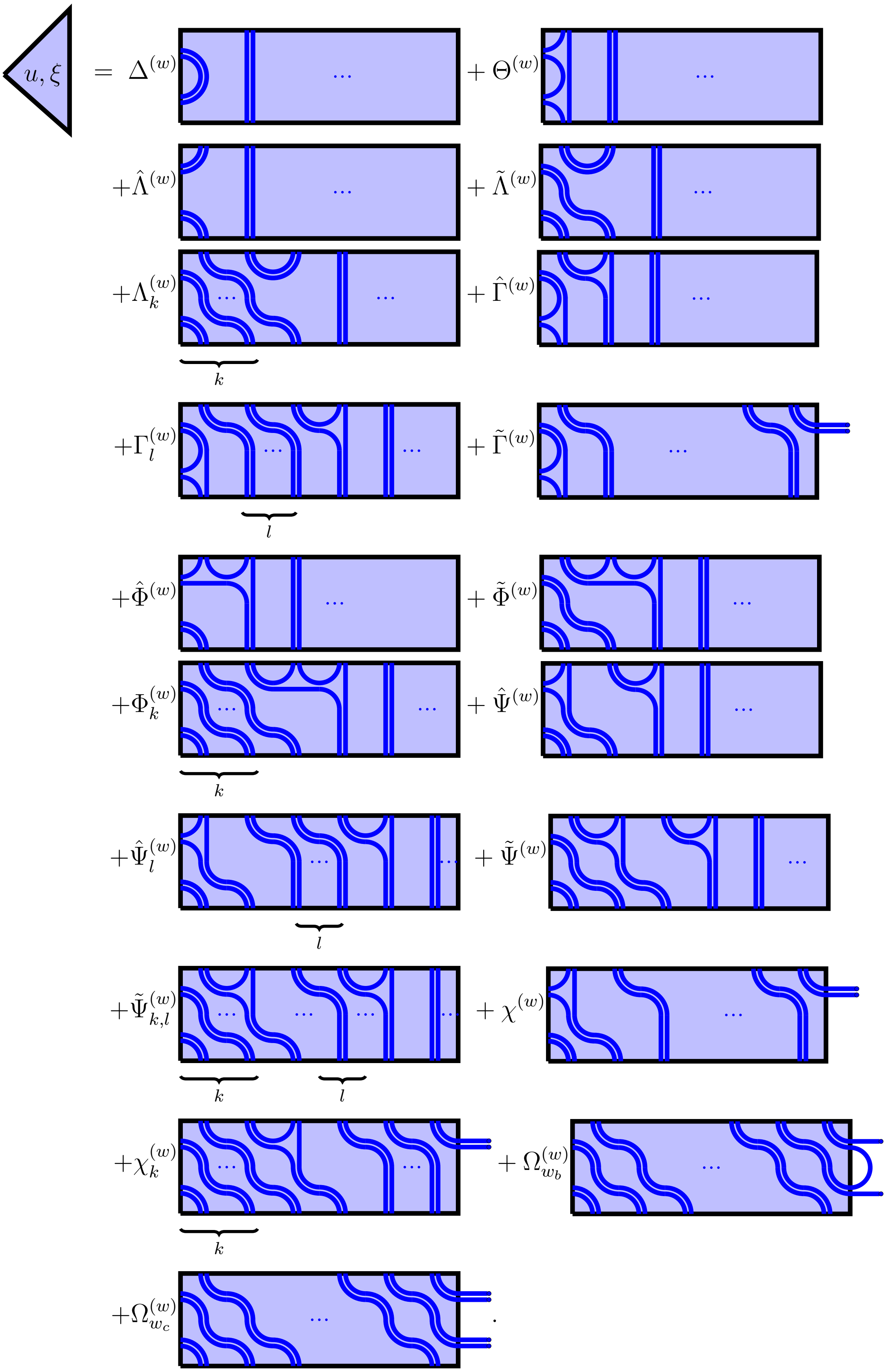


(4.21)



(4.22)

Here we see for $w > 4$, these diagrams evolve into the final reoccurring form expressed in equation (4.20). From the elementary forms expressed in equations (4.9) – (4.22) we can construct the objects



u, ξ
$= \Delta^{(w)}$
$+ \Theta^{(w)}$
$+\hat{\Lambda}^{(w)}$
$+ \tilde{\Lambda}^{(w)}$
$+\Lambda_k^{(w)}$
$+ \hat{\Gamma}^{(w)}$
k
$+\Gamma_l^{(w)}$
$+ \tilde{\Gamma}^{(w)}$
l
$+\hat{\Phi}^{(w)}$
$+ \tilde{\Phi}^{(w)}$
$+\Phi_k^{(w)}$
$+ \hat{\Psi}^{(w)}$
k
$+\hat{\Psi}_l^{(w)}$
$+ \tilde{\Psi}^{(w)}$
l
$+\tilde{\Psi}_{k,l}^{(w)}$
$+ \chi^{(w)}$
k
l
$+\chi_k^{(w)}$
$+ \Omega_{w_b}^{(w)}$
k
$+\Omega_{w_c}^{(w)}$
.

Again, we direct the interested reader to Appendix B.2 where the coefficients associated with each diagram are presented. The goal of this chapter was to extend the Robin boundary triangles, enriching the description of the model. In doing so we have developed an infinite class of integrable boundary conditions for the one-boundary (2,2) Temperley-Lieb loop model.

5

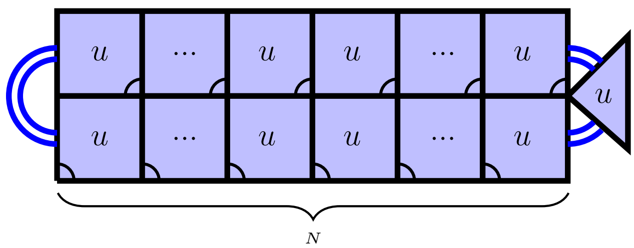
Hamiltonian Limit

The previous chapter established the integrability of the one-boundary (2,2) Temperley-Lieb loop model. We now seek to develop our understanding of this model. Of significant interest is the Hamiltonian, allowing us to determine the underlying energy level structure of the one-boundary (2,2) Temperley-Lieb loop model.

5.1 Double Row Transfer Tangle

In order to determine the Hamiltonian we must first introduce the double row transfer tangle, a diagrammatic object given by

$$D(u) = \frac{1}{\beta^2 - 1}$$



. (5.1)

The integrability of the one-boundary $(2,2)$ Temperley-Lieb loop model can be applied to construct a family of commuting transfer tangles

$$[D(u), D(v)] = 0. \tag{5.2}$$

The author would like to direct the interested reader to [27] for a through treatment of this proof. The iterative application of the transfer tangle on an arbitrary initial configuration $C_{(i)}$ of dimension $1 \times N$ constructs a lattice $C_{(f)}$ of dimension $2M \times N$, given by

$$C_{(f)} = [D(u)]^M C_{(i)}.$$

Here we can interpret $D(u)$ as an operator generating the lattice. Expanding on this notion, one can consider the dimension iterated by this operator as temporal rather than spacial $\Delta M \to \delta\tau$. Equipped with this interpretation we can associate the double row transfer tangle with the time translation operator of quantum mechanics

$$D = e^{\delta\tau H\kappa} \to U = e^{-itH}, \tag{5.3}$$

where κ defines a normalisation constant. Motivated by this association it is well established that the normalised double row transfer tangle admits an expansion of the form [1, 2]

$$D(u) = I - \frac{2u}{\sin\lambda} H + \mathcal{O}(u^2). \tag{5.4}$$

Such an expression resembles the power series expansion of the time translation operator introduced in equation (5.3). It follows that the expansion of this object will allow the direct determination of the one-boundary $(2, 2)$ Temperley-Lieb loop model Hamiltonian.

5.2 Hamiltonian Expansion

We now seek to take the Hamiltonian expansion of the double row transfer tangle equipped with the boundary conditions established in Chapters 3 and 4. For this treatment we will examine the case $w = 0$ where we are left with a single Robin boundary triangle. Performing the expansion to first order, we must take the first order series expansion of the Boltzmann weights associated with the $(2, 2)$ bulk and boundary elementary face operators. Due to the forms of both the YBE and the BYBE we are free to define an overall normalisation factor. This factor is assigned to ensure

$$\lim_{u\to 0} \hat{\alpha}_0(u) = 1 \qquad \text{and} \qquad \lim_{u\to 0} \hat{\alpha}_1(u) = \lim_{u\to 0} \hat{\alpha}_2(u) = 0,$$

where these expressions are the normalisation of the weights $\alpha_0^{2,2}(u)$, $\alpha_1^{2,2}(u)$ and $\alpha_0^{2,2}(u)$ respectively. Assigning the normalisation factor accordingly, we have the expressions

$$\hat{\alpha}_0(u) = -\frac{\sin(u-2\lambda)\sin^2(u-\lambda)}{\beta\sin^3\lambda}, \tag{5.5}$$

$$\hat{\alpha}_1(u) = \frac{\sin^2(u-\lambda)\sin(u)}{\sin^3\lambda}, \tag{5.6}$$

$$\hat{\alpha}_2(u) = -\frac{\sin(u)\sin(u+\lambda)\sin(u-\lambda)}{\beta\sin^3\lambda}. \tag{5.7}$$

Expanding these objects to first order we have

$$\hat{\alpha}_0(u) = 1 - \left(\frac{\cos(2\lambda) + \beta^2}{\beta \sin\lambda}\right) u + O(u^2), \tag{5.8}$$
$$\hat{\alpha}_1(u) = \frac{u}{\sin\lambda} + O(u^2), \tag{5.9}$$
$$\hat{\alpha}_2(u) = \frac{u}{\beta \sin\lambda} + O(u^2). \tag{5.10}$$

The normalisation of the boundary weights was developed in Chapter 3, see equations (3.20) – (3.20). Expanding these expressions to first order we have

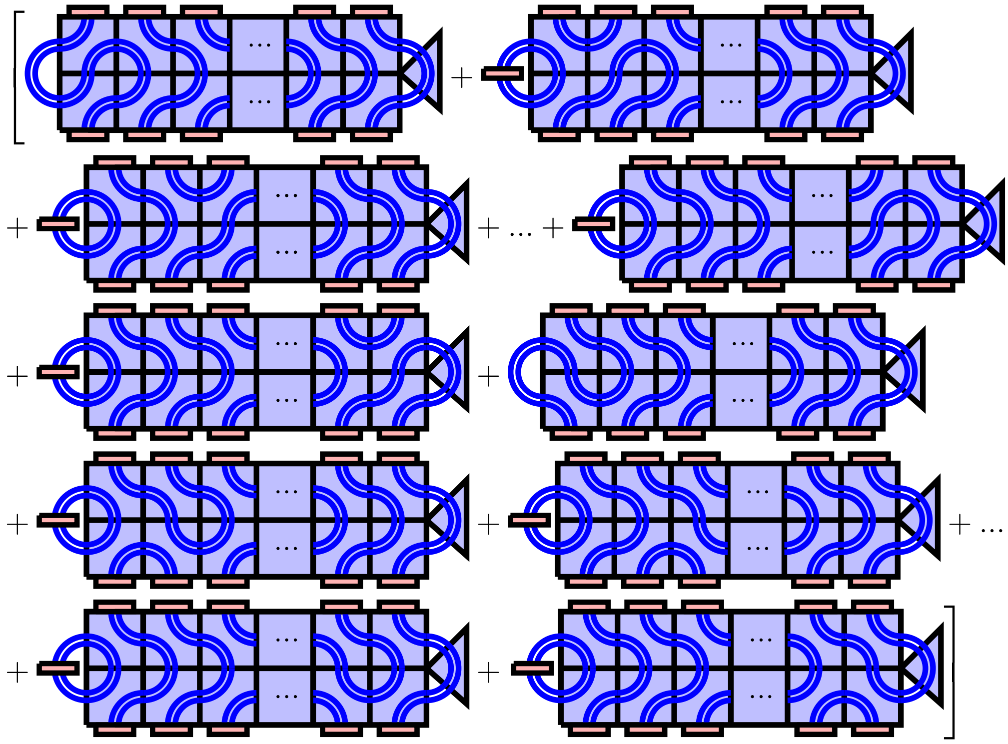
$$A_\eta(u) = 1 + \frac{u\left(\beta_2 \sin^3\lambda(2\beta_2\cos\lambda - \beta_1) - 2\eta^2 \cot\lambda\right)}{\eta\left(\eta - \beta_2 \sin^2\lambda\right)} + O\left(u^2\right), \tag{5.11}$$

$$B_\eta(u) = \frac{2u\sin\lambda\left(\sin^2\lambda(\beta_1 - \beta\beta_2) + \eta\beta\right)}{\eta\left(\eta - \beta_2\sin^2\lambda\right)} + O\left(u^2\right), \tag{5.12}$$

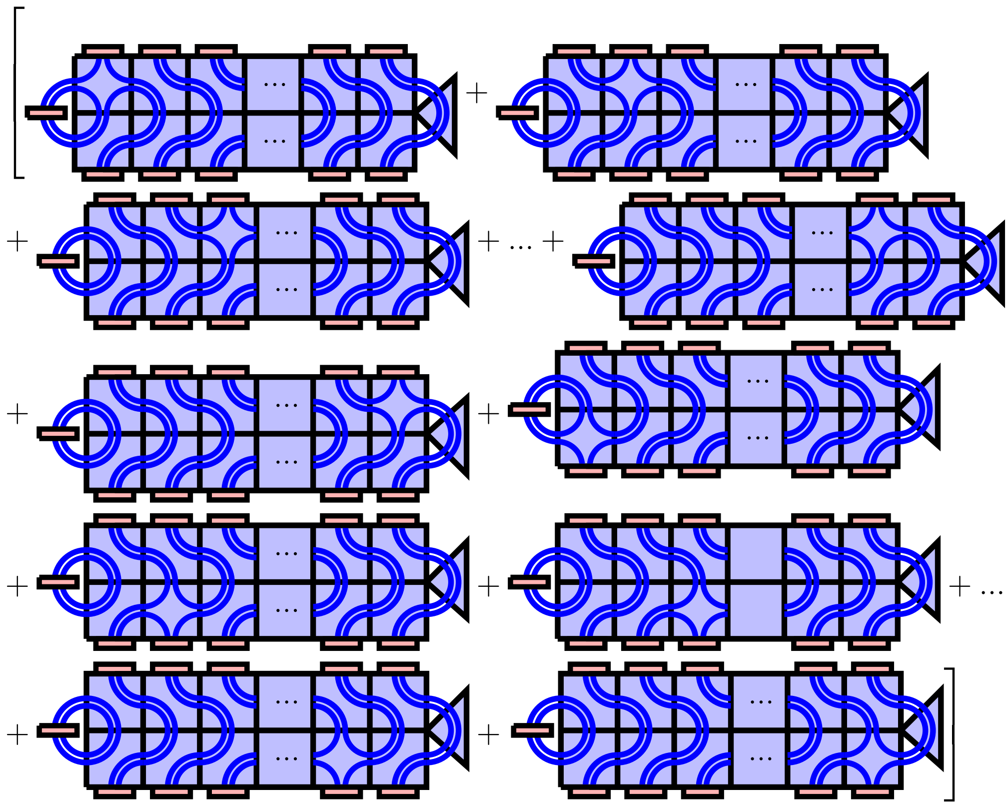
$$C_\eta(u) = -\frac{2u\sin^3\lambda}{\eta(\eta - \beta_2\sin^2\lambda)} + O\left(u^2\right). \tag{5.13}$$

To avoid confusion we will refer to the diagrammatic object associated with a particular coefficient $E(u)$ as simply E. Examining the expansions of $\hat{\alpha}_0(u)$ and $A_\eta(u)$,these are the only coefficients with zeroth order terms. The remaining coefficients only possess first order terms. Let us now consider the contributing diagrams at first order. Placing the object A_η on the boundary there exists $4N+1$ possible tangles. These are given by constructing the entire bulk from the orientation $\hat{\alpha}_0$ except for one position which can be taken by any of the remaining two bulk orientations $\hat{\alpha}_1$ or $\hat{\alpha}_2$, cycling these two objects through the $2N$ places produces $4N$ contributing tangles. The extra tangle is created by having the bulk consist of all orientations $\hat{\alpha}_0$ and the boundary with the A_η object. Now let us place the object B_η on the boundary. As $B_\eta(u)$ is first order in u, there exists no freedom to vary the bulk. Thus it must be constructed out of zeroth order terms given by the bulk orientation $\hat{\alpha}_0$. Finally, placing C_η on the boundary we have a similar result as for B_η. It follows that this expansion has $4N+3$ contributing terms at first order. Performing this expansion diagrammatically we have

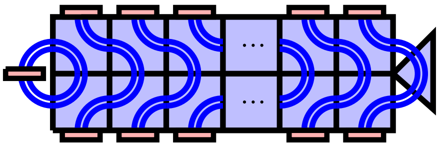
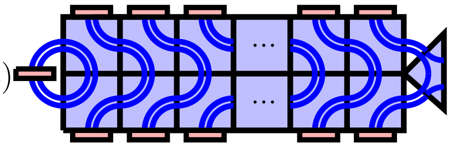
$$(\beta^2 - 1)D(u) = \hat{\alpha}_2[\hat{\alpha}_0]^{2N-1}A_\eta(u)$$

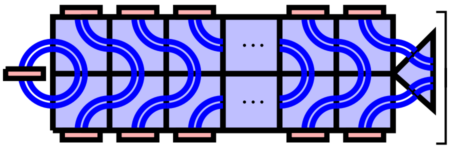


$$+\ \hat{\alpha}_1[\hat{\alpha}_0]^{2N-1}A_\eta(u)$$



$+ [\hat{\alpha}_0]^{2N}$

$\Big[A_\eta(u)$  $+ B_\eta(u)$ 

$+ C_\eta(u)$  $\Big] + \mathcal{O}(u^2).$

Replacing these objects with the generators of the algebra we have

$$
\begin{aligned}
(\beta^2-1)D(u) = {} & 2\hat{\alpha}_2[\hat{\alpha}_0]^{2N-1}A_\eta(u)\mathbb{I} + 2\hat{\alpha}_1[\hat{\alpha}_0]^{2N-1}A_\eta(u)\left(\beta - \frac{1}{\beta}\right)\mathbb{I} \\
& + 2\hat{\alpha}_2[\hat{\alpha}_0]^{2N-1}A_\eta(u)(\beta^2-1)[\mathbb{E}_1 + \mathbb{E}_2 + ... + \mathbb{E}_{N-2} + \mathbb{E}_{N-1}] \\
& + 2\hat{\alpha}_1[\hat{\alpha}_0]^{2N-1}A_\eta(u)(\beta^2-1)[\mathbb{X}_1 + \mathbb{X}_2 + ... + \mathbb{X}_{N-2} + \mathbb{X}_{N-1}] \\
& + [\hat{\alpha}_0]^{2N}[A_\eta\mathbb{I} + B_\eta(u)\mathbb{X}_N + C_\eta\mathbb{E}_N]\Big) + \mathcal{O}(u^2)
\end{aligned} \tag{5.14}
$$

$$
\begin{aligned}
D(u) = {} & \left(\frac{2\hat{\alpha}_2[\hat{\alpha}_0]^{2N-1}A_\eta(u)}{\beta^2-1} + 2\hat{\alpha}_1[\hat{\alpha}_0]^{2N-1}A_\eta(u)\frac{1}{\beta} + [\hat{\alpha}_0]^{2N}A_\eta\right)\mathbb{I} \\
& + 2\hat{\alpha}_2[\hat{\alpha}_0]^{2N-1}A_\eta(u)\sum_{j=1}^{N-1}\mathbb{E}_j + 2\hat{\alpha}_1[\hat{\alpha}_0]^{2N-1}A_\eta(u)\sum_{j=1}^{N-1}\mathbb{X}_j \\
& + [\hat{\alpha}_0]^{2N}[B_\eta(u)\mathbb{X}_N + C_\eta\mathbb{E}_N] + \mathcal{O}(u^2).
\end{aligned} \tag{5.15}
$$

These coefficients to first order are given by

$$\hat{\alpha}_2[\hat{\alpha}_0]^{2N-1}A_\eta(u) = \frac{u}{\beta\sin\lambda} + \mathcal{O}(u^2) \tag{5.16}$$

$$\hat{\alpha}_1[\hat{\alpha}_0]^{2N-1}A_\eta(u) = \frac{u}{\sin\lambda} + \mathcal{O}(u^2) \tag{5.17}$$

$$
\begin{aligned}
[\hat{\alpha}_0]^{2N}A_\eta(u) = {} & 1 - \left[\frac{\cos(2\lambda) + \beta^2}{\beta\sin\lambda}\right]u \\
& + \left[\frac{\left(\beta_2\sin^3\lambda(\beta\beta_2 - \beta_1) - 2\eta^2\cot\lambda\right)}{\eta\left(\eta - \beta_2\sin^2\lambda\right)}\right]u + O(u^2)
\end{aligned} \tag{5.18}
$$

$$[\hat{\alpha}_0]^{2N}B_\eta(u) = \left[\frac{2\sin\lambda\left(\sin^2\lambda(\beta_1 - \beta\beta_2) + \eta\beta\right)}{\eta\left(\eta - \beta_2\sin^2\lambda\right)}\right]u + O(u^2) \tag{5.19}$$

$$[\hat{\alpha}_0]^{2N}C_\eta(u) = -\left[\frac{2\sin^3\lambda}{\eta(\eta - \beta_2\sin^2\lambda)}\right]u + O(u^2). \tag{5.20}$$

Applying these expansions, we have

$$
\begin{aligned}
D(u) = &\left(\frac{2}{\beta^2 - 1} \frac{u}{\beta \sin\lambda} + \frac{2u}{\beta\sin\lambda} + 1 - \left[\frac{\cos(2\lambda) + \beta^2}{\beta\sin\lambda} \right] u + \left[\frac{\left(\beta_2 \sin^3\lambda(\beta\beta_2\lambda - \beta_1) - 2\eta^2 \cot\lambda\right)}{\eta\left(\eta - \beta_2\sin^2\lambda\right)} \right] u \right) \mathbb{I} \\
&+ \frac{2u}{\beta\sin\lambda} \sum_{j=1}^{N-1} \mathbb{E}_j + \frac{2u}{\sin\lambda} \sum_{j=1}^{N-1} \mathbb{X}_j + \left[\frac{2\sin\lambda\left(\sin^2\lambda(\beta_1 - \beta\beta_2) + \eta\beta\right)}{\eta\left(\eta - \beta_2\sin^2\lambda\right)} \right] u\mathbb{X}_N \\
&- \left[\frac{2u\sin^3\lambda}{\eta(\eta - \beta_2\sin^2\lambda)} \right] u\mathbb{E}_N + \mathcal{O}(u^2).
\end{aligned}
$$

Rearranging this expression

$$
\begin{aligned}
D(u) = \mathbb{I} - \frac{2u}{\sin\lambda} \Bigg(&\left(\frac{\cos(2\lambda) + \beta^2}{2\beta} - \frac{\beta}{\beta^2 - 1} - \frac{1}{\beta} - \left[\frac{\left(\beta_2 \sin^4\lambda(\beta\beta_2 - \beta_1) - 2\eta^2\cos\lambda\right)}{2\eta\left(\eta - \beta_2\sin^2\lambda\right)} \right] \right) I \\
&- \frac{1}{\beta} \sum_{j=1}^{N-1} \mathbb{E}_j - \sum_{j=1}^{N-1} \mathbb{X}_j - \left[\frac{\sin^2\lambda\left(\sin^2\lambda(\beta_1 - \beta\beta_2) + \eta\beta\right)}{\eta\left(\eta - \beta_2\sin^2\lambda\right)} \right] \mathbb{X}_N + \left[\frac{\sin^4\lambda}{\eta(\eta - \beta_2\sin^2\lambda)} \right] \mathbb{E}_N \Bigg) + \mathcal{O}(u^2).
\end{aligned}
$$

The Hamiltonian be can directly read as

$$
\begin{aligned}
H = &\left(\frac{\cos(2\lambda) + \beta^2}{2\beta} - \frac{\beta}{\beta^2 - 1} - \frac{1}{\beta} - \left[\frac{\left(\beta_2 \sin^4\lambda(\beta\beta_2 - \beta_1) - 2\eta^2\cos\lambda\right)}{2\eta\left(\eta - \beta_2\sin^2\lambda\right)} \right] \right) I \\
&- \frac{1}{\beta} \sum_{j=1}^{N-1} \mathbb{E}_j - \sum_{j=1}^{N-1} \mathbb{X}_j - \left[\frac{\sin^2\lambda\left(\sin^2\lambda(\beta_1 - \beta\beta_2) + \eta\beta\right)}{\eta\left(\eta - \beta_2\sin^2\lambda\right)} \right] \mathbb{X}_N + \left[\frac{\sin^4\lambda}{\eta(\eta - \beta_2\sin^2\lambda)} \right] \mathbb{E}_N \Bigg).
\end{aligned} \tag{5.21}
$$

Fixing the ground state energy contribution and setting the free parameter $\eta = \tilde{\eta}\ \sin^2\lambda$, produces the result

$$
\mathcal{H} = -\sum_{j=1}^{N-1} \left(\frac{1}{\beta}\mathbb{E}_j + \mathbb{X}_j \right) + \frac{1}{\tilde{\eta}(\tilde{\eta} - \beta_2)} \Big[\{\beta\beta_2 - \beta_1 - \tilde{\eta}\beta\}\mathbb{X}_N + \mathbb{E}_N \Big]. \tag{5.22}
$$

Here we have arrived at the Hamiltonian of the one-boundary $(2,2)$ Temperley-Lieb loop model. Examining the form of this expression we see there exist three possible poles for the parameter values $\tilde{\eta} = \beta_2$, $\tilde{\eta} = 0$ and $\beta = 0$. As $\tilde{\eta}$ and β_2 are free parameters of the model assigning appropriate values to these variables we will ensure the poles of the Hamiltonian are avoided. The parameter β is defined by λ as in equation (2.10) avoiding this pole we will enforce $\lambda \neq \frac{n\pi}{2}$ where $n \in \mathbb{Z}$.

6

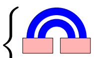
Representations

Chapter 5 concluded with an abstract algebraic description of the one-boundary (2,2) Temperley-Lieb Hamiltonian. A natural question arises, what is the energy spectrum of this object? To address this question we look to representation theory, allowing us to represent such objects as matrices. With this goal in mind we must first define the space on which these operators act.

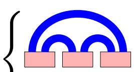
6.1 Link-states

The generators of the $(2,2)$ Temperley-Lieb algebra $\mathcal{TL}_N(\beta)$ act on a vector space of link-states V_N, also known as a linkspace. The link-states are analogous to a set of states in a quantum system. These states are non-intersecting diagrams that connect N pairs of nodes bound by projectors [28]. Let us examine the first two non-trivial linkspaces V_2 and V_3

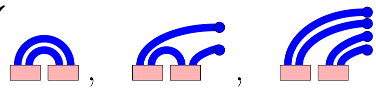
$$V_2 = \text{ span } \left\{ \quad \right\}, \tag{6.1}$$

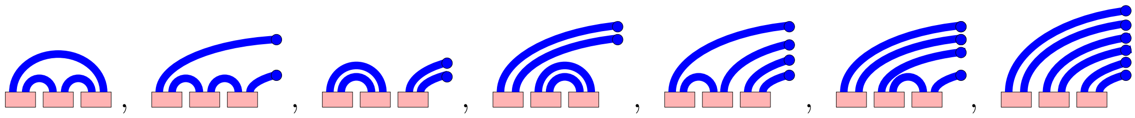


$$V_3 = \text{ span } \left\{ \quad \right\}. \tag{6.2}$$



Permitting connections to the boundary we define $\mathcal{V}_N^k$ as the space on which the generators of the one-boundary $(2,2)$ Temperley-Lieb algebra $\mathcal{BTL}_N(\beta;\beta_1,\beta_2)$ act. The index k labels the number of paired boundary connections. When the k index is omitted the space will consist of all possible boundary connections. Examining the spaces $\mathcal{V}_2$ and $\mathcal{V}_3$

$\mathcal{V}_2 = \text{span}\Big\{$  $\Big\}$, (6.3)

$\mathcal{V}_3 = \text{span}\Big\{$  $\Big\}$, (6.4)

we see that the extension to the boundary introduces two and six additional basis states to the $N = 2, 3$ linkspaces respectively. As is evident V_2 and V_3 are subspaces of $\mathcal{V}_2$ and $\mathcal{V}_3$ respectively ($V_2 \subset \mathcal{V}_2$, $V_3 \subset \mathcal{V}_3$). This is true for all N ($V_N \subset \mathcal{V}_N$). The dimension of the vector space $\mathcal{V}_N^k$ can be seen as equivalent to that established in [28]. This is evident by associating defects – straight lines beginning at a node, with boundary connections. The dimension of each space is given by the Riordan numbers $R_{N,k}$

$$\dim \mathcal{V}_N^k = R_{N,k} = \binom{N}{k}_2 - \binom{N}{k+1}_2, \tag{6.5}$$

where

$$\binom{N}{k}_2 = \sum_{j=0}^{N} \left[\begin{array}{c} N \\ \frac{1}{2}(N-j-k), \frac{1}{2}(N-j+k), j \end{array} \right],$$

and the trinomials are of the form

$$\left[\begin{array}{c} N \\ l, m, N-l-m \end{array} \right] = \begin{cases} \frac{N!}{l!m!(N-l-m)!}, & l, m, N-l-m \in \mathbb{Z}_{\geq 0} \\ 0, & \text{else} \end{cases}.$$

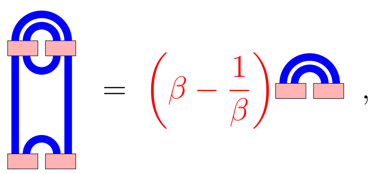
For convenience we have produced a table of relevant Riordan numbers, see table 6.1.

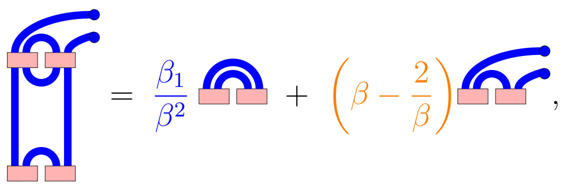
$N \backslash k$	0	1	2	3	4	5	6	7	8
1	0	1	0	0	0	0	0	0	0
2	1	1	1	0	0	0	0	0	0
3	1	3	2	1	0	0	0	0	0
4	3	6	6	3	1	0	0	0	0
5	6	15	15	10	4	1	0	0	0
6	15	36	40	29	15	5	1	0	0
7	36	91	105	84	49	21	6	1	0
8	91	232	280	238	154	76	28	7	1

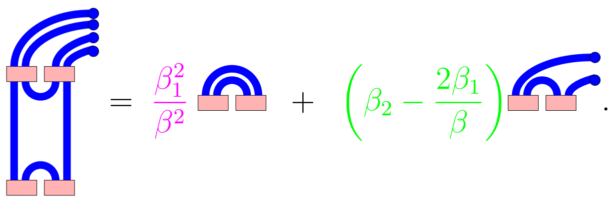
Table 6.1: Riordan numbers where N labels the system size and k counts the number of paired boundary connections or paired defects.

6.1.1 Link-state Representation

Let us now explore how the generators of our algebra act on the link-states. Similar to the definition of multiplication, the action of our operators on the set of link-states is given by vertical concatenation. As an example let us determine the matrix representation of the operator $\mathbb{X}_1 \in \mathcal{BTL}_2(\beta; \beta_1, \beta_2)$. These computations are performed by placing each link-state atop the generator $\mathbb{X}_1$ and evaluating the connectivities of the loop segments. This process is as follows







The matrix representation is now constructed by aligning the input states as columns and the resulting output states as rows. This convention is demonstrated by the coloration of the coefficients associated with each output link-state

$$\rho_2(\mathbb{X}_1) = \begin{pmatrix} \beta - \frac{1}{\beta} & \frac{\beta_1}{\beta^2} & \frac{\beta_1^2}{\beta^2} \\ 0 & \beta - \frac{2}{\beta} & \beta_2 - \frac{2\beta_1}{\beta} \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.6}$$

Here we have introduced the convention ρ_N as labelling the link-state representation of $\mathcal{BTL}_N(\beta;\beta_1,\beta_2)$. Applying this same process to the remaining generators of our algebra we have

$$\rho_2(\mathbb{E}_1) = \begin{pmatrix} \beta\left(\beta - \frac{1}{\beta}\right) & \beta_1\left(\beta - \frac{1}{\beta}\right) & \beta_1\left(\beta_2 - \frac{\beta_1}{\beta}\right) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.7}$$

$$\rho_2(\mathbb{E}_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \beta_2 - \frac{\beta_1}{\beta} & \beta_1(\beta_2 - \frac{\beta_1}{\beta}) \end{pmatrix}, \tag{6.8}$$

$$\rho_2(\mathbb{X}_1) = \begin{pmatrix} \beta - \frac{1}{\beta} & \frac{\beta_1}{\beta^2} & \frac{\beta_1^2}{\beta^2} \\ 0 & \beta - \frac{2}{\beta} & \beta_2 - \frac{2\beta_1}{\beta} \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.9}$$

$$\rho_2(\mathbb{X}_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \beta_2 & 0 \\ 0 & -\frac{1}{\beta} & \beta_2 - \frac{\beta_1}{\beta} \end{pmatrix}. \tag{6.10}$$

As we have developed how one represents the operators of our algebra as matrices, we are now in a position to explore the representation of the Hamiltonian – a special object within our algebra.

6.2 Hamiltonian Representations

Recall the one-boundary $(2,2)$ Temperley-Lieb Hamiltonian from equation (5.22). We seek to determine a representation of this object, that is

$$\rho_N(\mathcal{H}) = -\sum_{j=1}^{N-1}\left(\frac{1}{\beta}\rho_N(\mathbb{E}_j) + \rho_N(\mathbb{X}_j)\right) + \frac{1}{\tilde{\eta}(\tilde{\eta}-\beta_2)}\Big[\{\beta\beta_2 - \beta_1 - \tilde{\eta}\beta\}\rho_N(\mathbb{X}_N) + \rho_N(\mathbb{E}_N)\Big]. \tag{6.11}$$

Not only do we wish to explore the energy spectrum for a fixed N, we also seek to gain insight into how this spectrum varies as the system size N is increased. Ultimately one would like to examine the continuum limit of the system where $N \to \infty$; in this regime such information is necessary. Although interesting, such limits are beyond the scope of this thesis. Here we seek to provide a solid foundation from which arguments regarding the energy level structure can be made for large N.

Let us now determine the Hamiltonian for the first non-trivial algebra $\mathcal{BTL}_2(\beta;\beta_1,\beta_2)$. Applying the representation of the operators $\rho_2(\mathbb{E}_1)$, $\rho_2(\mathbb{E}_2)$, $\rho_2(\mathbb{X}_1)$ and $\rho_2(\mathbb{X}_2)$ from equations (6.7) – (6.10) to the Hamiltonian presented in equation (6.11), we arrive at a representation of $\mathcal{H} \in \mathcal{BTL}_2(\beta;\beta_1,\beta_2)$

$$\rho_2(\mathcal{H}) = \begin{pmatrix} \frac{2}{\beta} - 2\beta & -\frac{\left(\beta-\frac{1}{\beta}\right)\beta_1}{\beta} - \frac{\beta_1}{\beta^2} & -\frac{\beta_1^2}{\beta^2} - \frac{\left(\beta_2-\frac{\beta_1}{\beta}\right)\beta_1}{\beta} \\ -\frac{-\beta_1+\beta\beta_2-\beta\tilde{\eta}}{(\beta_2-\tilde{\eta})\tilde{\eta}} & -\beta - \frac{\beta_2(-\beta_1+\beta\beta_2-\beta\tilde{\eta})}{(\beta_2-\tilde{\eta})\tilde{\eta}} + \frac{2}{\beta} & \frac{2\beta_1}{\beta} - \beta_2 \\ -\frac{1}{(\beta_2-\tilde{\eta})\tilde{\eta}} & -\frac{-\frac{\beta_1}{\beta}+\beta_2-\frac{-\beta_1+\beta\beta_2-\beta\tilde{\eta}}{\beta}}{(\beta_2-\tilde{\eta})\tilde{\eta}} & -\frac{\beta_1\left(\beta_2-\frac{\beta_1}{\beta}\right)+(-\beta_1+\beta\beta_2-\beta\tilde{\eta})\left(\beta_2-\frac{\beta_1}{\beta}\right)}{(\beta_2-\tilde{\eta})\tilde{\eta}} \end{pmatrix}. \tag{6.12}$$

In order for this exercise to be computationally feasible we must fix the parameters $\beta, \beta_1, \beta_2, \tilde{\eta}$. Avoiding the poles identified in Chapter 5, the most natural designation is given by

$$\beta = \beta_1 = \beta_2 = 1 \qquad \text{and} \qquad \tilde{\eta} = 2. \tag{6.13}$$

The matrix representation simplifies to

$$\rho_2(\mathcal{H}) = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ \frac{1}{2} & 1 & 0 \end{pmatrix}. \tag{6.14}$$

The choice of the parameters detailed in equation (6.13) will be reevaluated in Section 6.2.2. Applying a similar approach to that presented in Section 6.1.1, one can compute the following Hamiltonian representations for $N = 3, 4, 5$

$$\rho_3(\mathcal{H}) = \begin{pmatrix} 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 & -1 & -1 & -1 \\ \frac{1}{2} & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \end{pmatrix}, \tag{6.15}$$

$$
\rho_4(\mathcal{H}) = \begin{pmatrix}
3 & -2 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 2 & -2 & -1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 \\
0 & \frac{1}{2} & 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & 0
\end{pmatrix}, \tag{6.16}
$$

$$
\rho_5(\mathcal{H}) = \left(\begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccc}
4 & -2 & -1 & -2 & -2 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & 1 & -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & -1 & -1 & -2 & -1 & -1 & -2 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 2 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & -1 & -1 & 0 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0
\end{array}\right). \tag{6.17}
$$

See Appendix C.1 for the form of each Hamiltonian in terms of the parameters β, β_1, β_2 and $\tilde{\eta}$. Illustrated in the growth of the matrix dimension; the basis states of the vector space $\mathcal{V}_N$ increase significantly for large N, see table 6.1. A major limitation of these calculation was the sheer time required to produce matrix representations of each individual generator. Examining the Riordan numbers, the next size up possesses $\dim\mathcal{V}_6 = 141$ link-states. A calculation of this size by hand is an inefficient use of resources. We envisage future work where a computer program is created that determines the matrix representation of any object within the algebra $\mathcal{BTL}_N(\beta;\beta_1,\beta_2)$. Such a program will provide the required information when the system is taken to the continuous limit.

6.2.1 Eigensystems

Now we seek to explore the energy level structure and the diagonalisability of the matrix representations presented in the previous section. For this initial examination the parameters remain fixed as defined by equation (6.13). We will adopt the indexing convention $\lambda^{(N)}$, characterising the set of eigenvalues with system size N

$$\lambda^{(2)} = \left\{\frac{1}{2}\left(\sqrt{3}+1\right), -1, \frac{1}{2}\left(1-\sqrt{3}\right)\right\},$$

$$\lambda^{(3)} = \{2.97848, 1.8055, -1.72644, -1, 1, 0.942463, 0\},$$

$$\begin{aligned}\lambda^{(4)} = \{&4.58575, 3.68124, -3.20147, 2.93518, 2.74657, -2.54433, 2.46962, 2.27718, -1.42336,\\ &-1.41421, 1.41421, 1.35159, 1.20336, 1.13674, -0.672982, 0.454904, 0, 0, 0\},\end{aligned}$$

$$\begin{aligned}\lambda^{(5)} = \{&6.19006, 5.4516, 4.80273, 4.54755, 4.42378, 4.21699, 3.76703, 3.70519, -3.7013, 3.51826,\\ &-3.50353, 3.43233, 3.34814, 2.87863, 2.64718, 2.64718, 2.64457, 2.62474, -2.54451, 2.30887,\\ &-2.26608, 2.16475, -2.12941, 2.07859, -2.03138, 2, 1.7282, 1.7282, 1.65902, 1.56963, -1.43442,\\ &1.35281, 1.32259, -1.30009, 1.05791, -1, -1, 0.797026, -0.756101, -0.756101, 0.673363, 0.615542,\\ &0.615542, -0.604208, 0.510353, 0.473028, -0.369377, 0.283841, -0.23483, -0.23483, 0.0808893\}.\end{aligned}$$

See Appendix C.3 for the corresponding eigenvectors. Examining the eigenvalues of the system we see they are real for all $N = 2, 3, 4, 5$, this is a remarkable result! Examining the representations of each Hamiltonian presented in equations (6.14) – (6.17) the entries are real however the matrix is highly non symmetric. It follows the matrix representations of the Hamiltonians are non-hermitian

$$\rho_N(\mathcal{H}) \neq \rho_N^{\dagger}(\mathcal{H}),$$

yet each spectra is physical. This result provides strong evidence for the following statements i) the procedures applied to ensure the integrability, determine the Hamiltonian and the corresponding representations are sound ii) the values assigned in equation (6.13) are sensible.

Another property of interest is that of diagonalisability, allowing one to express the Hamiltonian in the basis of it's eigenstates. Examining the eigensystems of $\rho_2(\mathcal{H})$ and $\rho_3(\mathcal{H})$ it follows these matrices are diagonalisable, admitting the forms

$$\mathcal{D}_2 = \mathrm{diag}(\lambda^{(2)}) \tag{6.18}$$

$$\mathcal{D}_3 = \mathrm{diag}(\lambda^{(3)}) \tag{6.19}$$

The remaining representations $\rho_4(\mathcal{H})$ and $\rho_5(\mathcal{H})$ are not diagonalisable. The next best thing is to determine a set of generalised eigenvectors. A description of the Hamiltonian in this basis will ensure it possesses Jordan normal form. Here we use the notation $J_n(\lambda)$, defining the Jordan block with dimension n associated with the eigenvalue λ. Omitting trivial blocks the Jordan normal forms of these Hamiltonians are given by

$$\mathcal{J}_4 = \mathrm{diag}(J_3(0)), \tag{6.20}$$

$$\mathcal{J}_5 = \mathrm{diag}(J_2(2.6472), J_2(1.7282), J_2(-1), J_2(-0.7561), J_2(0.6155), J_2(-0.2346)). \tag{6.21}$$

Examining each Jordan normal form. $\mathcal{J}_4$ possesses a single non-trivial Jordan block of dimension 3 associated with the eigenvalue 0 and $\mathcal{J}_5$ possesses six non-trivial Jordan blocks of dimension 2 associated with the eigenvalues $2.6472, 1.7282, -1, -0.7561, 0.6155$ and -0.2346.

6.2.2 Varying Parameters

The previous section explored the eigensystem of each Hamiltonian representation for the parameter values $\beta, \beta_1, \beta_2, \tilde{\eta}$ specified in equation (6.13). In this section we seek to provide a preliminary glance at another set of 'natural' values for these parameters. Ideally this analysis should provide further insight into the nature of the eigenvalues and the conditions of diagonalisability. Divergences existing in the Hamiltonian place constraints on the parameters $\tilde{\eta}$ and β. Ensuring the poles associated with $\tilde{\eta}$ are avoided we will set this parameter to $\tilde{\eta} = 2$, for the remaining investigation. This assignment will allow us to explore the influence of the loop fugacities on the eigensystem.

The bulk loop fugacity $\beta = 2\cos\lambda$, is the only parameter possessing a known closed form expression. Rich results arise in the representation theory of the $(1,1)$ Temperley-Lieb algebra in cases when β can be described by roots of unity [29, 30]. Values of λ giving rise to such cases are given by the set

$$\lambda \in \left\{ \frac{n\pi}{2}, \frac{n\pi}{3}, \frac{n\pi}{4}, \frac{n\pi}{6}, 0 \right\}, \qquad n \in \mathbb{Z} \tag{6.22}$$

corresponding to the set of β values,

$$|\beta| \in \left\{0, 1, \sqrt{2}, \sqrt{3}, 2\right\}. \tag{6.23}$$

These values will form the basis of 'natural' values that can be assigned to the loop fugacities β, β_1 and β_2. In this preliminary investigation we examine each of the possible combinations of the loop fugacities possessing the values 1 and $\sqrt{2}$. We seek to examine both the reality of the spectrum and the diagonalisability of the matrix representation associated with the Hamiltonian for each of the loop fugacity combinations. Representing this information in a succinct manner we have constructed tables and used the conventions: real (R), a non-zero imaginary component (C), diagonalisable (D), non-diagonalisable (N-D). Computational limitations prevented the diagonalisability of the matrices $N = 4, 5$ being determined.

$N=2$	β	β_1	β_2
R, D	1	1	1
R, D	$\sqrt{2}$	1	1
R, D	1	$\sqrt{2}$	1
C, D	1	1	$\sqrt{2}$
R, D	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
R, D	1	$\sqrt{2}$	$\sqrt{2}$
R, D	$\sqrt{2}$	1	$\sqrt{2}$
R, D	$\sqrt{2}$	$\sqrt{2}$	1

$N=3$	β	β_1	β_2
R, D	1	1	1
C, D	$\sqrt{2}$	1	1
R, D	1	$\sqrt{2}$	1
C, D	1	1	$\sqrt{2}$
R, D	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
R, D	1	$\sqrt{2}$	$\sqrt{2}$
C, D	$\sqrt{2}$	1	$\sqrt{2}$
R, D	$\sqrt{2}$	$\sqrt{2}$	1

$N=4$	β	β_1	β_2
R	1	1	1
C	$\sqrt{2}$	1	1
R	1	$\sqrt{2}$	1
C	1	1	$\sqrt{2}$
R	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
R	1	$\sqrt{2}$	$\sqrt{2}$
C	$\sqrt{2}$	1	$\sqrt{2}$
R	$\sqrt{2}$	$\sqrt{2}$	1

$N=5$	β	β_1	β_2
R	1	1	1
C	$\sqrt{2}$	1	1
C	1	$\sqrt{2}$	1
C	1	1	$\sqrt{2}$
C	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
C	1	$\sqrt{2}$	$\sqrt{2}$
C	$\sqrt{2}$	1	$\sqrt{2}$
R	$\sqrt{2}$	$\sqrt{2}$	1

TABLE 6.2: Evaluating the influence of the parameters β, β_1 and β_2, nine different tuples are constructed and applied to the Hamiltonian representations. The reality of the spectrum for the Hamiltonian representations are evaluated for system sizes $N = 2, 3, 4, 5$ and the diagonalisability for $N = 2, 3$. We use the conventions real (R), a non-zero imaginary component (C), diagonalisable (D), non-diagonalisable (N-D).

Analysing the spectra we see that as the system size increases so does the instances of complexity. Contrasting $N = 2$ with $N = 5$ the former possesses seven out of eight real parameter tuples whereas the latter only possesses two out of eight. Interestingly the nature of the spectrum remained the same for $N = 3$ and $N = 4$.

There exist two tuples $\{1,\ 1\,,1\}$ and $\{\sqrt{2},\ \sqrt{2}\,,1\}$ that remain real throughout the data set. We associate these with the most natural parameter designations corresponding to physical systems. The first tuple is unsurprising and was used in our initial analysis performed in Section 6.2.1. The second is interesting, demonstrating the asymmetry between the parameters β_1 and β_2. In order to extract more information regarding the influence of these parameters on the reality of the spectra we require a deeper examination of the influence of varying the parameter $\tilde{\eta}$. An investigation of this nature will form a future direction of the project.

Examining the diagonalisability, it is evident there is limited data available. This is a consequence of the matrix dimensions for $N = 4, 5$. Of the $N = 2, 3$ tuples analysed all are diagonalisable. In addition to further exploration of the parameter space we seek to employ greater computational power to examine the diagonalisability of larger system sizes.

6.3 Invariant Bilinear Form

Our focus now turns toward the representation theory of the $(2, 2)$ Temperley-Lieb algebra $\mathcal{TL}_N(\beta)$, specifically the question of irreducibility. In quantum mechanics irreducible representations of the symmetry group associated with a Hamiltonian characterise the corresponding eigenspace, hence providing a consistent state labelling system [31]. Fuelled by this observation, we seek to implement a similar approach as in [29]; examining an invariant bilinear form on the link-states of $\mathcal{TL}_N(\beta)$. As we will demonstrate, the conditions of irreducibility for the $(1, 1)$ and the $(2, 2)$ Temperley-Lieb algebras are strikingly similar. As far as the author is aware an examination of this nature for $\mathcal{TL}_N(\beta)$ has not been presented in the literature and hence forms a novel result of this thesis.

An invariant bilinear form on the vector space of linkstates V_N is a map

$$\langle\ ,\ \rangle : V_N \times V_N \to \mathbb{C}.$$

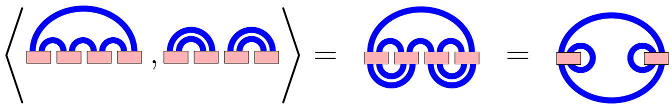
Bilinearity ensures it must satisfy the following relations

$$\langle av, bw\rangle = ab\langle v, w\rangle, \tag{6.24}$$

$$\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_1, w\rangle, \tag{6.25}$$

$$\langle v, w_1 + w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle, \tag{6.26}$$

where $v, v_1, v_2, w, w_1, w_2 \in V_N$ $a, b \in \mathbb{C}$. For the Temperley-Lieb algebra we calculate the bilinear form by placing the link-state of the first argument in the upright position, the second argument is then reflected about the horizontal and the nodes of each diagram are matched in a consistent manner. This action is demonstrated below

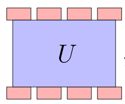


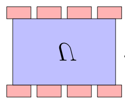
$$= \beta\left(\beta - \frac{1}{\beta}\right)^2, \qquad (6.27)$$

where in the second equality we exploited the property of projectors $P_2^2 = P_2$. An interesting property of the bilinear form is given by

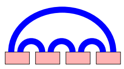
$$\langle Uv, w\rangle = \langle v, U^\dagger w\rangle, \qquad \forall\, U,\ U^\dagger \in \mathcal{TL}_N(\beta) \qquad (6.28)$$

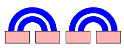
Here we have introduced the complex conjugate $U^\dagger$ of an object U in $\mathcal{TL}_N(\beta)$. This operation is simply the mirror image of the original object about the horizontal [29], diagrammatically

$U =$  , (6.29)

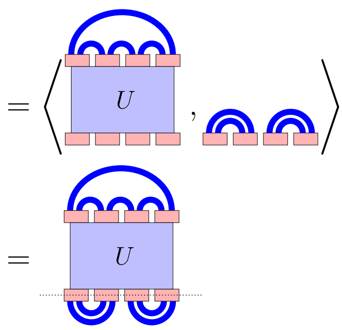
$U^\dagger =$  . (6.30)

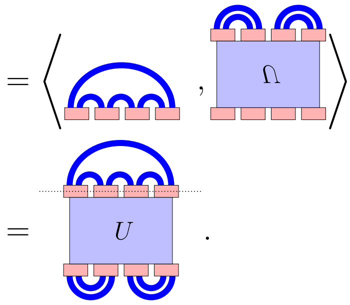
Let us show equation (6.28) holds for an arbitrary configuration U and the hermitian conjugate $U^\dagger$. Without loss of generality we assign the link-states

$v =$  , (6.31)

$w =$  . (6.32)

Applying these objects to each side of equation (6.28) we have

$\langle Uv, w\rangle =$ 

$\langle v, U^\dagger w\rangle =$  .

Here the dotted line denotes the axis about which the second argument was reflected. Examining the forms of each expansion we see they are identical. Equating these terms, equation (6.28) directly follows. Having established the properties of the bilinear form we can discuss how it provides insight into the irreducibility of V_N. The radical of this bilinear form on V_N is defined

$$R_N = \{x \in V_N | \ \langle x, y\rangle = 0, \quad \forall y \in V_N\}. \tag{6.33}$$

Proposition 3.3 from [29] states, the radical R_N is itself a submodule of V_N. It follows that if a space V_N possesses a non-trivial radical this link space is reducible. To examine the radical of a linkspace we require one final ingredient, the *Gram matrix*.

6.3.1 Gram Matrix

The Gram matrix facilitates the systematic analysis of the bilinear product of each link-state within V_N. This object is defined

$$\mathcal{G}_N = \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_2, v_1\rangle & \dots & \langle v_m, v_1\rangle \\ \langle v_1, v_2\rangle & \langle v_2, v_2\rangle & \dots & \langle v_m, v_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_1, v_m\rangle & \langle v_2, v_m\rangle & \dots & \langle v_m, v_m\rangle \end{pmatrix}, \tag{6.34}$$

where $v_1, v_2, ..., v_m \in V_N$ and $m = \text{dim}V_N$. Of great interest is the determinant of the Gram matrix. Indeed, if $\det\mathcal{G}_N = 0$ it follows there exists a non-trivial radical implying the reducibility of V_N. We seek to develop a closed form expression for the determinant of the Gram matrix associated with the linkspace V_N. This expression provides insight into the conditions of irreducibility for any linkspace V_N. To achieve this, the Gram matrices for the first six non-trivial linkspaces ($N = 2, 3, 4, 5, 6, 7$) have been determined. The first four are given by

$$\mathcal{G}_2 = \beta\left(\beta - \frac{1}{\beta}\right) \tag{6.35}$$

$$\mathcal{G}_3 = \beta\left(\beta - \frac{1}{\beta}\right)\left(\beta - \frac{2}{\beta}\right) \tag{6.36}$$

$$\mathcal{G}_4 = \begin{pmatrix} \beta(\beta-\frac{1}{\beta})[(\beta-\frac{1}{\beta})(\beta-\frac{2}{\beta})+\frac{1}{\beta^2}] & \beta(\beta-\frac{1}{\beta})^2 & \beta(\beta-\frac{1}{\beta})^2 \\ \beta(\beta-\frac{1}{\beta})^2 & \beta^2(\beta-\frac{1}{\beta})^2 & \beta(\beta-\frac{1}{\beta}) \\ \beta(\beta-\frac{1}{\beta})^2 & \beta(\beta-\frac{1}{\beta}) & \beta^2(\beta-\frac{1}{\beta})^2 \end{pmatrix} \tag{6.37}$$

$$\mathcal{G}_5 = \begin{pmatrix} \beta(\beta-\frac{1}{\beta})(\beta-\frac{2}{\beta})\left[(\beta-\frac{1}{\beta})(\beta-\frac{2}{\beta})+1\right] & \beta(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) & \beta(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) & \beta(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) & \beta(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) & \beta(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) \\ \beta(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) & \beta^2(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) & \beta(\beta-\frac{1}{\beta})(\beta-\frac{2}{\beta}) & 0 & 0 & \beta(\beta-\frac{1}{\beta})(\beta-\frac{2}{\beta}) \\ \beta(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) & \beta(\beta-\frac{1}{\beta})(\beta-\frac{2}{\beta}) & \beta^2(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) & \beta(\beta-\frac{1}{\beta})(\beta-\frac{2}{\beta}) & 0 & 0 \\ \beta(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) & 0 & \beta(\beta-\frac{1}{\beta})(\beta-\frac{2}{\beta}) & \beta^2(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) & \beta(\beta-\frac{1}{\beta})(\beta-\frac{2}{\beta}) & 0 \\ \beta(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) & 0 & 0 & \beta(\beta-\frac{1}{\beta})(\beta-\frac{2}{\beta}) & \beta^2(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) & \beta(\beta-\frac{1}{\beta})(\beta-\frac{2}{\beta}) \\ \beta(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) & \beta(\beta-\frac{1}{\beta})(\beta-\frac{2}{\beta}) & 0 & 0 & \beta(\beta-\frac{1}{\beta})(\beta-\frac{2}{\beta}) & \beta^2(\beta-\frac{1}{\beta})^2(\beta-\frac{2}{\beta}) \end{pmatrix}. \tag{6.38}$$

The first two are one dimensional matrices as the linkspaces possess only one basis state. Examining the remaining forms we see these matrices are symmetric, this follows from the symmetry of the bilinear product. The matrices $\mathcal{G}_6$ and $\mathcal{G}_7$ are too large to display, see Appendix C.2 for the *Mathematica* code to construct these objects. Taking the determinant of these matrices we have the sequence

$$\begin{aligned}
\det(\mathcal{G}_2) =&\beta^2 - 1\\
\det(\mathcal{G}_3) =&\frac{1}{\beta^2}(\beta^2 - 1)(\beta^3 - 2\beta)\\
\det(\mathcal{G}_4) =&\frac{1}{\beta^4}\left(\beta^2 - 1\right)^3\left(\beta^3 - 2\beta\right)^2\left(\beta^4 - 3\beta^2 + 1\right)\\
\det(\mathcal{G}_5) =&\frac{1}{\beta^{15}}(\beta^2 - 1)^5(\beta^3 - 2\beta)^6(\beta^4 - 3\beta^2 + 1)^3(\beta^5 - 4\beta^3 + 3\beta)\\
\det(\mathcal{G}_6) =&\frac{1}{\beta^{40}}(\beta^2 - 1)^{11}(\beta^3 - 2\beta)^{14}(\beta^4 - 3\beta^2 + 1)^{10}(\beta^5 - 4\beta^3 + 3\beta)^4(\beta^6 - 5\beta^4 + 6\beta^2 - 1)\\
\det(\mathcal{G}_7) =&\frac{1}{\beta^{119}}(\beta^2 - 1)^{21}(\beta^3 - 2\beta)^{35}(\beta^4 - 3\beta^2 + 1)^{28}(\beta^5 - 4\beta^3 + 3\beta)^{15}(\beta^6 - 5\beta^4 + 6\beta^2 - 1)^5\\
&\quad(\beta^7 - 6\beta^5 + 10\beta^3 - 4\beta)^1.
\end{aligned} \tag{6.39}$$

Examining this sequence we see Chebyshev polynomials of the second kind arising naturally. These polynomials are defined recursively by

$$\begin{aligned}
U_0(\beta/2) &= 1\\
U_1(\beta/2) &= \beta\\
U_{n+1}(\beta/2) &= \beta U_n(\beta/2) - U_{n-1}(\beta/2).
\end{aligned} \tag{6.40}$$

We will need the first seven polynomials, for convenience these are listed as follows

$$\begin{aligned}
U_0 &= 1\\
U_1 &= \beta\\
U_2 &= \beta^2 - 1\\
U_3 &= \beta^3 - 2\beta\\
U_4 &= \beta^4 - 3\beta^2 + 1\\
U_5 &= \beta^5 - 4\beta^2 + 3\beta\\
U_6 &= \beta^6 - 5\beta^4 + 6\beta^2 - 1\\
U_7 &= \beta^7 - 6\beta^5 + 10\beta^3 - 4\beta.
\end{aligned} \tag{6.41}$$

Here we have used the convention $U_n = U_n(\beta/2)$. Expressing the sequence of Gram determinants in terms of Chebyshev polynomials, we have

$$
\begin{aligned}
\det(\mathcal{G}_2) &= \left(\frac{1}{\beta}\right)^0 \left(\frac{U_1}{U_0}\right)^1 \left(\frac{U_2}{U_1}\right)^1 \\
\det(\mathcal{G}_3) &= \left(\frac{1}{\beta}\right)^3 \left(\frac{U_1}{U_0}\right)^3 \left(\frac{U_2}{U_1}\right)^2 \left(\frac{U_3}{U_2}\right)^1 \\
\det(\mathcal{G}_4) &= \left(\frac{1}{\beta}\right)^4 \left(\frac{U_1}{U_0}\right)^6 \left(\frac{U_2}{U_1}\right)^6 \left(\frac{U_3}{U_2}\right)^3 \left(\frac{U_4}{U_3}\right)^1 \\
\det(\mathcal{G}_5) &= \left(\frac{1}{\beta}\right)^{15} \left(\frac{U_1}{U_0}\right)^{15} \left(\frac{U_2}{U_1}\right)^{15} \left(\frac{U_3}{U_2}\right)^{10} \left(\frac{U_4}{U_3}\right)^4 \left(\frac{U_5}{U_4}\right)^1 \\
\det(\mathcal{G}_6) &= \left(\frac{1}{\beta}\right)^{36} \left(\frac{U_1}{U_0}\right)^{36} \left(\frac{U_2}{U_1}\right)^{40} \left(\frac{U_3}{U_2}\right)^{29} \left(\frac{U_4}{U_3}\right)^{15} \left(\frac{U_5}{U_4}\right)^5 \left(\frac{U_6}{U_5}\right)^1 \\
\det(\mathcal{G}_7) &= \left(\frac{1}{\beta}\right)^{105} \left(\frac{U_1}{U_0}\right)^{91} \left(\frac{U_2}{U_1}\right)^{105} \left(\frac{U_3}{U_2}\right)^{84} \left(\frac{U_4}{U_3}\right)^{49} \left(\frac{U_5}{U_4}\right)^{21} \left(\frac{U_6}{U_5}\right)^6 \left(\frac{U_7}{U_6}\right)^1 .
\end{aligned}
\tag{6.42}
$$

Examining this sequence we propose the following closed form expression for $\det(\mathcal{G}_N)$

$$
\det(\mathcal{G}_N) = \left(\frac{U_0}{U_1}\right)^{N\,\mathrm{dim}V^0_{N-1}} \prod_{j=1}^{N} \left(\frac{U_j}{U_{j-1}}\right)^{\mathrm{dim}V^j_N} . \tag{6.43}
$$

Comparing this result to that of the $(1,1)$ Temperley-Lieb algebra $\det(G_N)$, presented in [29]. This equation given by

$$
\det(G_N) = \prod_{l=1}^{N} \left(\frac{U_l}{U_{l-1}}\right)^{\mathrm{dim}\overline{V}^{2l}_N} . \tag{6.44}
$$

The author would like to express astonishment at the striking similarity between the forms of the two expressions. Note that $\mathrm{dim}V^j_N$ and $\mathrm{dim}\overline{V}^{2l}_N$ give the dimensions of the vector spaces associated with the $(2,2)$ Temperley-Lieb algebra $\mathcal{TL}_N(\beta)$ and the $(1,1)$ Temperley-Lieb algebra $TL_N(\beta)$ algebra respectively. Furthermore j defines a cabled boundary connection whereas l defines a single boundary connection, it follows $j = 2l$. As in the $TL_N(\beta)$ case, the conditions of irreducibility of the $\mathcal{TL}_N(\beta)$ linkspace effectively reduces to an examination of Chebyshev polynomial zeros.

6.3.2 Conditions of Irreducibility

This section will conclude our examination of the $(2,2)$ Temperley-Lieb algebra representation theory. Here we examine the conditions on the parameter β ensuring an irreducible representation. Recall the form of the loop fugacity β

$$
\beta = 2\cos\lambda. \tag{6.45}
$$

The Chebyshev polynomials associated with the Gram determinant are given by $U_x(\beta/2)$. It suffices to examine the zeros of this function over the domain $[-1, 1]$. As in [32], this polynomial can be expressed in terms of trigonometric functions

$$U_x(\beta/2) = \frac{\sin\left((x+1)\arccos(\beta/2)\right)}{\sin\left(\arccos(\beta/2)\right)}. \tag{6.46}$$

Solving for the roots of this expression it follows that they are real, distinct and are given by the function

$$\beta = 2\cos\left(\left(\frac{k}{x+1}\right)\pi\right), \qquad k = 1, ..., x. \tag{6.47}$$

Here we have found the roots of Chebyshev polynomial. Let us now take a closer look at the determinant of the Gram matrix as to avoid the poles of this function. The Gram determinant $\det(\mathcal{G}_N)$, is presented in equation (6.43) and can be re-expressed as the proportionality

$$\det(\mathcal{G}_N) \propto \prod_{j=1}^{N} U_j(\beta/2)^{\mathrm{dim}V_N^j - \mathrm{dim}V_N^{j+1}}. \tag{6.48}$$

This expression details the essential information required for the analysis of the conditions of irreducibility. Examining the exponent

$$\Delta\mathrm{dim}V_N^j = \mathrm{dim}V_N^j - \mathrm{dim}V_N^{j+1} = \binom{N}{j}_2 + \binom{N}{j+2}_2, \tag{6.49}$$

here we have simply applied the definition of Riordan numbers, see equation (6.5). We are particularly interested in the cases where $\Delta\mathrm{dim}V_N^j < 0$. In such instances the zeros of the Chebyshev polynomials of order $1 \leq j$ will become poles of the determinant. We propose two approaches providing “approximate” conditions on the irreducibility of a particular linkspace.

Approach I: *Avoid the poles*

We have an understanding of both the zeros and poles of the Gram determinant in terms of the Chebyshev polynomials, see equations (6.47) and (6.48). Avoiding both the zeros and the poles of the Gram determinant the conditions of irreducibility are given by

$$\beta \neq 2\cos\left(\left(\frac{k}{x+1}\right)\pi\right), \qquad k = 1, ..., x, \qquad x = 1, ..., N. \tag{6.50}$$

We recognise this condition is overly restrictive on the parameter β. The elimination of the poles will permit the form of β possessing values $x \leq j$, this case will now be explored.

Approach II: *Eliminate the poles*

Exploiting the factorisation properties of the Chebyshev polynomial developed in [32]

$$U_m(x) = U_h(x)\left(2\sum_{k=0}^{l}\left(T_{m-(2k+1)h-2k}(x) - 1\right)\right), \tag{6.51}$$

where

$$m = (2l+1)h + 2l, \qquad m, h, l \in \mathbb{Z}_+,$$

and $T_n(x)$ denotes the nth Chebyshev polynomials of the first kind. The second kind Chebyshev polynomials of order $x \leq j$ existing on the denominator can be cancelled by the factored forms of second kind Chebyshev polynomials of order $j+1 \leq x \leq m$. Here m defines the order of Chebyshev polynomials required to cancel those existing in the denominator. Eliminating the Chebyshev polynomials of order $x \leq j$ ensures these functions no longer contribute poles or zeros to the Gram determinant. It follows that β can be expressed in terms of zeros of these polynomials as they no longer exist in the Gram determinant. Given this consideration, the refined conditions of irreducibility are given by

$$\beta \neq 2\cos\left(\left(\frac{k}{x+1}\right)\pi\right), \qquad k = 1, ..., x, \qquad x = j+1, ..., N. \tag{6.52}$$

The second kind Chebyshev polynomials of order $j+1 \leq x \leq m$ were used to cancel those of order $x \leq j$, one can imagine the possibility of this cancellation process eliminating particular order polynomials on that interval. For those particular x, β is permitted to possess those values. Furthermore the introduction of the first order Chebyshev terms may introduce further zeros, requiring additional conditions be enforced on the parameter β.

The author would like to concede the crudity of these arguments however, a full examination of the conditions of irreducibility is beyond the scope of this thesis. Approach II is the most rigorous in determining the full conditions on the parameter β. Here we outline the steps required to achieve a thorough treatment of the irreducibility conditions. Understand the conditions on j and N when $\Delta \mathrm{dim} V_N^j < 0$. Determine an expression for m detailing the polynomials required to cancel the poles. Finally, examine the first order Chebyshev polynomial terms contributing new zeros to the Gram determinant.

7
Conclusion

Beginning with a brief introduction to loop models at large, this thesis has provided a focused look at the $(1,1)$ Temperley-Lieb loop model [13]. Integrable Robin boundary conditions for the $(1,1)$ model were presented and the underlying algebraic structure was examined. Following [2], a fusion procedure was detailed allowing the construction of a (m,n) Temperley-Lieb loop model from the elementary $(1,1)$ face operators. Particular attention was placed on the $(2,2)$ Temperley-Lieb loop model where we introduced generalised Robin boundary conditions. A solution to the BYBE was presented ensuring the $(2,2)$ Temperley-Lieb loop model remains integrable with the extension to the boundary. This solution was then used to construct a $(2,2)$ boundary operator from $(1,1)$ constituent operators. Following these advancements, the generators of the underlying one-boundary $(2,2)$ Temperley-Lieb algebra were listed and the defining relations were determined.

Applying a seam of width w, the Robin boundary conditions introduced to the $(2,2)$ Temperley-Lieb loop model were then used to construct an infinite class of integrable boundary conditions. Focus was then placed on determining the Hamiltonian of the $(2,2)$ Temperley-Lieb loop model with width $w = 0$. The integrability of the boundary conditions permitted the construction of a commuting family of transfer tangles. Taking the expansion of the transfer tangle allowed the Hamiltonian of the model to be determined. Exploring the representation theory of the one-boundary $(2,2)$ Temperley-Lieb algebra $\mathcal{BTL}_N(\beta;\beta_1,\beta_2)$ allowed the determination of the spectra for $N = 2,3,4,5$. Examining the resulting spectra for the aforementioned N, we provided insight into the energy level structure as the system size increases.

Our attention then turned toward the representation theory of the $(2,2)$ Temperley-Lieb algebra $\mathcal{TL}_N(\beta)$. Here an invariant bilinear form was constructed to examine the conditions of irreducibility on the linkspace associated with this algebra. A Gram matrix was constructed facilitating the systematic analysis of this form. Examining the first six non-trivial Gram matrices, a closed form expression of the determinant was proposed. Preliminary analysis of this object yields novel insights into the conditions of irreducibility of linkspace associated with the $(2,2)$ Temperley-Lieb algebra $\mathcal{TL}_N(\beta)$.

The key contributions of this thesis are the introduction of an infinite class of Yang-Baxter integrable boundary conditions to the $(2,2)$ Temperley-Lieb model and the novel closed form expression of the Gram matrix determinant for the linkspace associated with $\mathcal{TL}_N(\beta)$. These contributions motivate two distinct directions of future investigation.

Firstly, we seek to establish the drop-down properties associated with the extended boundary conditions, followed by a rigorous definition of the boundary Wenzl-Jones projector. Next, we envisage the generalisation of the fusion procedure developed for the $(2,2)$ boundary operator to an arbitrary size (n,n). Pairing this construction with the (n,n) bulk square operator introduced in Chapter 3, one may be able to construct an integrable one-boundary (n,n) Temperley-Lieb loop model. The energy level structure of the one-boundary $(2,2)$ Temperley-Lieb loop model was explored through matrix representations of the Hamiltonian associated with this model. It was recognised for particular parameter values the spectra was real despite the non-Hermiticity of these representations. A similar phenomena was observed for the Hamiltonian of the $(1,1)$ Temperley-Lieb loop model [15]. An argument was constructed by Morin-Duchesne *et al.* [33] demonstrating this result to be the consequence of the pseudo-Hermiticity of the Hamiltonian. We seek to adapt this argument to the one-boundary $(2,2)$ Temperley-Lieb Hamiltonian, providing a consistent interpretation of the reality of the spectra. The exploration of the energy level structure was conducted for small system sizes N. Developing a computer program to determine representations of the Hamiltonian for arbitrary system sizes will permit the examination of the energy level structure for larger N. This information will allow us to generalise from finite N to the continuous limit.

Secondly, we seek to continue our exploration of the Temperley-Lieb algebra representation theory. Of immediate interest is finalising the conditions on β ensuring the irreducibility of the linkspace V_N, associated with the $(2,2)$ Temperley-Lieb algebra $\mathcal{TL}_N(\beta)$. It follows that a similar methodology can be applied to the linkspace of the one-boundary $(2,2)$ Temperley-Lieb algebra $\mathcal{BTL}_N(\beta;\beta_1,\beta_2)$. An examination of this nature allows a determination of the conditions on the parameters β, β_1 and β_2 ensuring the irreducibility of the linkspace $\mathcal{V}_N$.

A
Boundary Yang Baxter Equation

A.1 Mathematica Code

Here we provide the code to verify the solution to the BYBE. Throughout the *Mathematica* code blocks we have used the conventions

$$\begin{aligned} F(v,\lambda) &= F[v,l] \\ \beta &= b[l] \\ \beta_1 &= b2 \\ \beta_2 &= b1 \\ \eta &= n \end{aligned}$$

A.1.1 Ratio

Here we present the code verifying the forms of $\mathcal{X}_\eta(v)$ and $\mathcal{Y}_\eta(v)$ are solutions to the BYBE

```
X[v_, l_] := (b2 + Y[v, l])((n + Sin[v] (b1 Sin[v - 2 l]
- b2 Sin[v - l]))/(b[l] Sin[l] Sin[2 v]))

Y[v_, l_] := ((-n b[l] + (b[l] b1 - b2) (Sin[l])^2 + (2 b2
- b[l] b1) (Sin[v])^2)/(Sin[l] Sin[2 v + l]))

b[l_]:= 2 Cos[l]

C1[u_, v_, l_] := 2 b2 b[l] Sin[l] Sin[v] Cos[v] X[v, l]
```

```
- 2 b2 b[l] Sin[l] Sin[u] Cos[u] X[u, l] + b2 Sin[u - v]
(-b2 Sin[u + v - l] + b1 Sin[u + v - 2 l]) Y[v, l] +
b2 Sin[u - v] (-b2 Sin[u + v - l] + b1 Sin[u + v - 2 l])
Y[u, l] + 2 b[l] Sin[l] Sin[v] Cos[v] X[v, l] Y[u, l] -
2 b[l] Sin[l] Sin[u] Cos[u] Y[v, l] X[u, l] + Sin[u - v]
(-b2 Sin[u + v - l] + b1 Sin[u + v - 2 l]) Y[v, l] Y[u, l]
+ b2^2 Sin[u - v] (-b2 Sin[u + v - l] + b1 Sin[u + v - 2 l])

C2[u_, v_, l_] := -(b1 b[l] - 2 b2) Sin[l] Sin[u + v] Sin[u - v]
Sin[2 v] X[v, l] - Sin[u + v] Sin[
    u - v] (b1 Sin[u + v - l] - b2 Sin[u + v]) (b1 Sin[u - v + l]
    - b2 Sin[u - v]) Y[v, l] - (Sin[l])^2 Sin[2 v] Sin[2 u + l]
    X[v, l] Y[u, l] + (Sin[l])^2 Sin[2 u] Sin[2 u + l] Y[v, l]
    X[u, l] - Sin[l] Sin[u - v] Sin[2 u + l] (b1 Sin[u + v - l]
    - b2 Sin[u + v]) Y[v, l] Y[u, l]

C3[u_, v_, l_] := -b[l] Sin[l] Sin[2 v] X[v, l] + b[l] Sin[l] Sin[2 u]
X[u, l] + (-b[l] Sin[l] (b1 - b2/b[l]) Sin[2 v] + ((2 b2)/b[l] - b1)
Sin[u + v] Sin[u - v]) Y[v, l] + (b[l] Sin[l] (b1 - b2/b[l]) Sin[2 u]
+ ((2 b2)/b[l] - b1) Sin[u + v] Sin[u - v]) Y[u, l] - 1/b[l] Sin[l]
(Sin[2 u + l] - Sin[2 v + l]) Y[v, l] Y[u, l] + (b[l] (b1 - b2/b[l])
Sin[u - v] (b1 Sin[u + v] - b2 Sin[u + v - l]) + b2 ((2 b2)/b[l] - b1)
Sin[u + v] Sin[u - v])

C4[u_, v_, l_] :=
 b[l] Sin[l] Sin[2 v] (Cos[2 l] Cos[2 v] - Cos[l] Cos[2 u - l])
 X[v, l] + b[l] (Sin[l])^2 Sin[2 u] Sin[2 u + l] X[u, l] +
 Sin[u + v - l] Sin[u - v] ((b[l] b1 - b2) Sin[u - v + l]
 Sin[u + v - l] - b2 Sin[u + v] Sin[u - v]) Y[v, l] + Sin[l]
 Sin[u - v] Sin[2 u + l] (-(b1 b[l] - 2 b2) Sin[u + v - l]
 + b2 Sin[u + v + l]) Y[u, l] + Sin[l] Sin[u + v - l] Sin[u - v]
 Sin[2 u + l] Y[v, l] Y[u, l] + (-b[l] b1^2 Sin[u + v - l]
 Sin[u + v] Sin[u - v] Sin[u - v + l] + b2 b1 b[l] Sin[u - v]
 (Sin[u + v - l] (Cos[l] Cos[2 v - l] - Cos[2 u]) +
 (Sin[u + v])^2 Sin[u - v + l]) - b2^2 Sin[u - v]
 ((Sin[u + v - l])^2 Sin[u - v + l] + Sin[u + v]
 Sin[u - v] (2 Sin[u + v - l] + Sin[u + v + l])))


C5[u_, v_, l_] := Sin[l] Sin[2 v + l] Y[v, l] - Sin[l] Sin[2 u + l]
Y[u, l] - (b[l] b1 - 2 b2) Sin[u - v] Sin[u + v]


  Simplify[C1[u, v, l]]
```

```
Simplify[C2[u, v, l]]

Simplify[C3[u, v, l]]

Simplify[C4[u, v, l]]

Simplify[C5[u, v, l]]
```

A.1.2 Coefficients

Here we present the code verifying the forms of $A_\eta(v)$, $B_\eta(v)$ and $C_\eta(v)$ are solutions to the BYBE

```
Ae[v_,l_]:= (Sin[l]/(n(n-b1 (Sin[l])^2)Sin[2v+l]))(n-b2 Sin[v+l]Sin[v]
+ b1 Sin[v+l]Sin[v-l])(n-b2 Sin[v]Sin[v-l] + b1 Sin[v]Sin[v-2l])

Be[v_,l_]:= (((Sin[l])^2 Sin[2v])/(n(b1 (Sin[l])^2-n)Sin[2v+l]))
((b[l]b1 - b2)(Sin[l])^2 - n b[l] + (2b2 - b[l] b1)(Sin[v])^2)

Ce[v_,l_]:= ((Sin[l])^3 Sin[2v])/(n(b1 (Sin[l])^2-n))

b[l_]:= 2 Cos[l]

D1[u_, v_,l_]:= 2 b2 b[l] Sin[l] Sin[v] Cos[v] Ae[v,l]Ce[u,l]-2 b2 b[l]
Sin[l] Sin[u] Cos[u] Ae[u,l]Ce[v,l]+b2 Sin[u-v](-b2 Sin[u+v-l]+
b1 Sin[u+v-2l]) Be[v,l]Ce[u,l]+b2 Sin[u-v] (-b2 Sin[u+v-l]+b1
Sin[u+v-2 l]) Be[u,l]Ce[v,l]+2 b[l] Sin[l] Sin[v] Cos[v] Ae[v,l] Be[u,l]
-2 b[l] Sin[l] Sin[u] Cos[u] Be[v,l] Ae[u,l]+Sin[u-v] (-b2 Sin[u+v-l]
+b1Sin[u+v-2 l])Be[v,l] Be[u,l]+b2^2 Sin[u-v]
(-b2 Sin[u+v-l]+b1 Sin[u+v-2 l])Ce[v,l] Ce[u,l]

D2[u_,v_,l_]:= -(b1 b[l]-2 b2) Sin[l]Sin[u+v] Sin[u-v] Sin[2v] Ae[v,l]Ce[u,l]
-Sin[u+v] Sin[u-v](b1 Sin[u+v-l]-b2 Sin[u+v]) (b1 Sin[u-v+l]-b2 Sin[u-v])
Be[v,l]Ce[u,l]-(Sin[l])^2 Sin[2v] Sin[2u+l] Ae[v,l]Be[u,l]+(Sin[l])^2 Sin[2u]
Sin[2u+l] Be[v,l] Ae[u,l]-Sin[l] Sin[u-v] Sin[2u+l] (b1 Sin[u+v-l]-b2 Sin[u+v])
Be[v,l] Be[u,l]

D3[u_,v_,l_]:= -b[l] Sin[l] Sin[2 v] Ae[v,l]Ce[u,l]+b[l] Sin[l] Sin[2 u]
Ae[u,l]Ce[v,l] +(-b[l] Sin[l](b1-b2/b[l]) Sin[2 v]+((2b2)/b[l]-b1) Sin[u+v]
Sin[u-v]) Be[v,l]Ce[u,l]+(b[l] Sin[l] (b1-b2/b[l]) Sin[2 u]+((2b2)/b[l]-b1)
Sin[u+v] Sin[u-v]) Be[u,l]Ce[v,l]-1/b[l] Sin[l] (Sin[2 u+l]-Sin[2 v+l])
Be[v,l] Be[u,l]+(b[l] (b1-b2/b[l]) Sin[u-v](b1 Sin[u+v]-b2 Sin[u+v-l])
+b2 ((2b2)/b[l]-b1) Sin[u+v] Sin[u-v])Ce[u,l]Ce[v,l]
```

```
D4[u_,v_,l_]:= b[l] Sin[l] Sin[2v] (Cos[2l] Cos[2v]-Cos[l] Cos[2u-l]) Ae[v,l]Ce[u,l]
+b[l] (Sin[l])^2 Sin[2u] Sin[2u+l] Ae[u,l]Ce[v,l]+Sin[u+v-l] Sin[u-v] ((b[l] b1-b2)
Sin[u-v+l] Sin[u+v-l]-b2 Sin[u+v] Sin[u-v]) Be[v,l]Ce[u,l]+Sin[l] Sin[u-v] Sin[2u+l]
(-(b1 b[l]-2 b2) Sin[u+v-l]+b2 Sin[u+v+l]) Be[u,l]Ce[v,l]+Sin[l] Sin[u+v-l] Sin[u-v]
Sin[2u+l] Be[v,l] Be[u,l]+(-b[l] b1^2 Sin[u+v-l] Sin[u+v] Sin[u-v] Sin[u-v+l]+
b2 b1 b[l] Sin[u-v] (Sin[u+v-l] (Cos[l] Cos[2v-l]-Cos[2u])+(Sin[u+v])^2 Sin[u-v+l])
-b2^2 Sin[u-v] ((Sin[u+v-l])^2 Sin[u-v+l]+Sin[u+v]Sin[u-v]
(2Sin[u+v-l]+Sin[u+v+l])))Ce[u,l]Ce[v,l]

D5[u_,v_,l_]:= Sin[l] Sin[2 v+l]Be[v,l]Ce[u,l]-Sin[l] Sin[2 u+l]Be[u,l]Ce[v,l]
-(b[l] b1-2 b2) Sin[u-v] Sin[u+v]Ce[u,l]Ce[v,l]


  Simplify[D1[u, v, l]]

  Simplify[D2[u, v, l]]

  Simplify[D3[u, v, l]]

  Simplify[D4[u, v, l]]

  Simplify[D5[u, v, l]]
```

B

Extended Boundary Coefficients

The size and number of coefficients for both the reoccuring forms and the boundary operators were far too large to be presented eloquently in the body of the report. As such we will present them in the following sections.

The extended boundary coefficients are defined in terms of the functions $\hat{\alpha}_0(u), \hat{\alpha}_1(u), \hat{\alpha}_2(u)$ and $A_\eta(u), B_\eta(u), C_\eta(u)$. For convenience these are listed below

$$
\begin{aligned}
\hat{\alpha}_0(u) &= -\frac{\sin(u-2\lambda)\sin^2(u-\lambda)}{\beta\sin^3\lambda} \\
\hat{\alpha}_1(u) &= \frac{\sin^2(u-\lambda)\sin(u)}{\sin^3\lambda} \\
\hat{\alpha}_2(u) &= -\frac{\sin(u)\sin(u+\lambda)\sin(u-\lambda)}{\beta\sin^3\lambda}.
\end{aligned}
$$

and

$$
A_\eta(v) = \left[\frac{\sin(v+\lambda)}{\eta\beta_2\sin\lambda\sin(2v+\lambda)}\right]\left[\beta_1\sin(v)-\beta_2\sin(v-\lambda)\right]\left[\eta+\sin(v)\left\{\beta_2\sin(v-2\lambda)-\beta_1\sin(v-\lambda)\right\}\right]
$$

$$
B_\eta(v) = \left[\frac{\sin(2v)}{\eta\beta_2\sin(2v+\lambda)}\right]\left[(\beta\beta_2-\beta_1)\sin^2\lambda-\eta\beta+(2\beta_1-\beta\beta_2)\sin^2(v)\right]
$$

$$
C_\eta(v) = \frac{\sin\lambda\sin(2v)}{\eta\beta_2}
$$

B.1 Reoccuring Forms

For a seam of arbitrary width $w \in 2\mathbb{N}_0$ we recognise 17 reoccurring forms. These are given by the coefficients $a^{(w)}(u,\xi_w)$ to $s^{(w)}(u,\xi_w)$, where the width of the seam is indexed by the superscript.

$$
\begin{aligned}
a^{(w)}(u,\xi_w) &= \hat{\alpha}_0(u+\xi_w)\hat{\alpha}_0(u-\xi_w)\\
b^{(w)}(u,\xi_w) &= \hat{\alpha}_1(u+\xi_w)\hat{\alpha}_0(u-\xi_w) + \hat{\alpha}_0(u+\xi_w)\hat{\alpha}_1(u-\xi_w) + \left(\beta - \frac{2}{\beta}\right)\hat{\alpha}_1(u+\xi_w)\hat{\alpha}_1(u-\xi_w)\\
c^{(w)}(u,\xi_w) &= \hat{\alpha}_2(u+\xi_w)\hat{\alpha}_0(u-\xi_w) + \frac{1}{\beta^2}\hat{\alpha}_1(u+\xi_w)\hat{\alpha}_1(u-\xi_w) + \hat{\alpha}_0(u+\xi_w)\hat{\alpha}_2(u-\xi_w)\\
&\quad + \left(\beta - \frac{1}{\beta}\right)\left[\hat{\alpha}_1(u+\xi_w)\hat{\alpha}_2(u-\xi_w) + \hat{\alpha}_2(u+\xi_w)\hat{\alpha}_1(u-\xi_w) + \beta\hat{\alpha}_2(u+\xi_w)\hat{\alpha}_2(u-\xi_w)\right]\\
d^{(w)}(u,\xi_w) &= -\frac{1}{\beta}\hat{\alpha}_0(u+\xi_w)\hat{\alpha}_1(u-\xi_w)\\
e^{(w)}(u,\xi_w) &= \frac{1}{\beta^2}\hat{\alpha}_1(u+\xi_w)\hat{\alpha}_1(u-\xi_w)\\
f^{(w)}(u,\xi_w) &= \frac{1}{\beta^2}\hat{\alpha}_1(u+\xi_w)\hat{\alpha}_1(u-\xi_w) + \left(\beta - \frac{2}{\beta}\right)\hat{\alpha}_1(u+\xi_w)\hat{\alpha}_2(u-\xi_w) + \hat{\alpha}_0(u+\xi_w)\hat{\alpha}_2(u-\xi_w)\\
g^{(w)}(u,\xi_w) &= \frac{1}{\beta^2}\left(\hat{\alpha}_2(u+\xi_w)\hat{\alpha}_1(u-\xi_w) + \hat{\alpha}_1(u+\xi_w)\hat{\alpha}_2(u-\xi_w) - \frac{1}{\beta}\hat{\alpha}_1(u+\xi_w)\hat{\alpha}_1(u-\xi_w)\right)\\
&\quad + \left(\beta - \frac{1}{\beta}\right)\hat{\alpha}_2(u+\xi_w)\hat{\alpha}_2(u-\xi_w)\\
h^{(w)}(u,\xi_w) &= \hat{\alpha}_2(u+\xi_w)\hat{\alpha}_2(u-\xi_w)\\
i^{(w)}(u,\xi_w) &= \hat{\alpha}_1(u+\xi_w)\hat{\alpha}_2(u-\xi_w)\\
j^{(w)}(u,\xi_w) &= \hat{\alpha}_0(u+\xi_w)\hat{\alpha}_2(u-\xi_w)\\
k^{(w)}(u,\xi_w) &= -\frac{1}{\beta}\hat{\alpha}_0(u+\xi_w)\hat{\alpha}_1(u-\xi_w)\\
l^{(w)}(u,\xi_w) &= \frac{1}{\beta^2}\hat{\alpha}_1(u+\xi_w)\hat{\alpha}_1(u-\xi_w) + \hat{\alpha}_0(u+\xi_w)\hat{\alpha}_2(u-\xi_w) + \left(\beta - \frac{2}{\beta}\right)\hat{\alpha}_1(u+\xi_w)\hat{\alpha}_2(u-\xi_w)\\
m^{(w)}(u,\xi_w) &= -\frac{1}{\beta}\hat{\alpha}_1(u+\xi_w)\hat{\alpha}_2(u-\xi_w)\\
n^{(w)}(u,\xi_w) &= \hat{\alpha}_0(u+\xi_w)\hat{\alpha}_2(u-\xi_w)\\
o^{(w)}(u,\xi_w) &= \hat{\alpha}_0(u+\xi_w)\hat{\alpha}_2(u-\xi_w)\\
p^{(w)}(u,\xi_w) &= -\frac{1}{\beta}\hat{\alpha}_1(u+\xi_w)\hat{\alpha}_2(u-\xi_w)\\
q^{(w)}(u,\xi_w) &= \hat{\alpha}_0(u+\xi_w)\hat{\alpha}_2(u-\xi_w)\\
r^{(w)}(u,\xi_w) &= -\frac{1}{\beta}\hat{\alpha}_0(u+\xi_w)\hat{\alpha}_1(u-\xi_w)
\end{aligned}
$$

$$s^{(w)}(u,\xi_w) = \frac{1}{\beta^2}\hat{\alpha}_1(u+\xi_w)\hat{\alpha}_1(u-\xi_w) + \hat{\alpha}_0(u+\xi_w)\hat{\alpha}_2(u-\xi_w) + \left(\beta - \frac{2}{\beta}\right)\hat{\alpha}_1(u+\xi_w)\hat{\alpha}_2(u-\xi_w)$$

Forms specific to small system sizes $w = 2,\ 4$ are given by

$$\begin{aligned}
t^{(2)}(u,\xi_2) &= \frac{\beta_1}{\beta^2}\hat{\alpha}_1(u+\xi_2)\hat{\alpha}_1(u-\xi_2)\\
u^{(2)}(u,\xi_2) &= \frac{\beta_1}{\beta^2}\left(\hat{\alpha}_2(u+\xi_2)\hat{\alpha}_1(u-\xi_2) + \hat{\alpha}_1(u+\xi_2)\hat{\alpha}_2(u-\xi_2) - \frac{1}{\beta}\hat{\alpha}_1(u+\xi_2)\hat{\alpha}_2(u-\xi_2)\right)\\
v^{(2)}(u,\xi_2) &= -\frac{\beta_1}{\beta}\hat{\alpha}_0(u+\xi_2)\hat{\alpha}_1(u-\xi_2)\\
w^{(2)}(u,\xi_2) &= \frac{\beta_1}{\beta^2}\hat{\alpha}_1(u+\xi_2)\hat{\alpha}_1(u-\xi_2) + \left(\beta_2 - \frac{2\beta_1}{\beta}\right)\hat{\alpha}_1(u+\xi_2)\hat{\alpha}_2(u-\xi_2)\\
x^{(2)}(u,\xi_2) &= \hat{\alpha}_0(u+\xi_2)\hat{\alpha}_2(u-\xi_2)\\
y^{(2)}(u,\xi_2) &= \frac{\beta_1^2}{\beta^2}\hat{\alpha}_1(u+\xi_2)\hat{\alpha}_1(u-\xi_2)\\
z^{(2)}(u,\xi_2) &= \frac{\beta_1^2}{\beta^2}\left(\hat{\alpha}_2(u+\xi_2)\hat{\alpha}_1(u-\xi_2) + \hat{\alpha}_1(u+\xi_2)\hat{\alpha}_2(u-\xi_2) - \frac{1}{\beta}\hat{\alpha}_1(u+\xi_2)\hat{\alpha}_1(u-\xi_2)\right)\\
&\quad + \beta_1\left(\beta_2 = \frac{\beta_1}{\beta}\right)\\
aa^{(4)}(u,\xi_4) &= \hat{\alpha}_0(u+\xi_4)\hat{\alpha}_2(u-\xi_4)\\
ab^{(4)}(u,\xi_4) &= \hat{\alpha}_0(u+\xi_4)\hat{\alpha}_2(u-\xi_4).
\end{aligned}$$

B.2 Boundary Operators

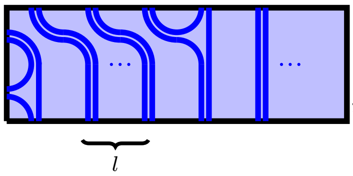
The boundary operators are constructed by the iterative application of the reoccurring forms defined above. Let us first present the base coefficients, these are given by the coefficients of the first non-trivial seam width $w = 2$

$$\begin{aligned}
\Delta^{(2)} =& a^{(2)}(u,\xi_2)A_\eta(u)\\
\Theta^{(2)} =& b^{(2)}(u,\xi_2)A_\eta(u) + t^{(2)}(u,\xi_2)B_\eta(u) + y^{(2)}(u,\xi_2)C_\eta(u)\\
\Lambda_0^{(2)} =& c^{(2)}(u,\xi_2)A_\eta(u) + u^{(2)}(u,\xi_2)B_\eta(u) + z^{(2)}(u,\xi_2)C_\eta(u)
\end{aligned}$$

For a general seam width the coefficients are defined recursively as follows. Note: we use the convention where $\prod_{(p=0)}^{N}$ is incremented by 2.

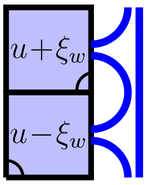
$$\begin{aligned}
\Delta^{(w)} =& a^{(w)}(u,\xi_w)\Delta^{(w-2)} \\
\Theta^{(w)} =& b^{(w)}(u,\xi_w)\Delta^{(w-2)} + e^{(w)}(u,\xi_w)\Theta^{(w-2)} \\
\hat{\Lambda}^{(w)} =& c^{(w)}(u,\xi_w)\Delta^{(w-2)} + g^{(w)}(u,\xi_w)\Theta^{(w-2)} + h^{(w)}(u,\xi_w)\hat{\Lambda}^{(w-2)} \\
\tilde{\Lambda}^{(w)} =& j^{(w)}(u,\xi_w)\hat{\Lambda}^{(w-2)} \\
\Lambda_k^{(w)} =& \Big(\prod_{(p=w-k+2)}^{w} q^{(p)}(u,\xi_p)\Big)\tilde{\Lambda}^{(w-k)} \\
\hat{\Gamma}^{(w)} =& d^{(w)}(u,\xi_w)\Theta^{(w-2)} \\
\Gamma_l^{(w)} =& \Big(\prod_{(p=w-l+2)}^{w} k^{(p)}(u,\xi_p)\Big)\hat{\Gamma}^{(w-l)} \\
\tilde{\Gamma}^{(w)} =& \Big(\prod_{(p=4)}^{w} k^{(p)}(u,\xi_p)\Big)(r^{(2)}(u,\xi_2)B_\eta(u) + v^{(2)}(u,\xi_2)C_\eta(u)) \\
\hat{\Phi}^{(w)} =& f^{(w)}(u,\xi_w)\Theta^{(w-2)} + i^{(w)}(u,\xi_w)\hat{\Lambda}^{(w-2)} \\
\tilde{\Phi}^{(w)} =& o^{(w)}(u,\xi_w)\hat{\Phi}^{(w-2)} \\
\Phi_k^{(w)} =& \Big(\prod_{(p=w-k+2)}^{w} q^{(p)}(u,\xi_p)\Big)\tilde{\Phi}^{(w-k)} \\
\hat{\Psi}^{(w)} =& p^{(w)}(u,\xi_w)\hat{\Phi}^{(w-2)} \\
\hat{\Psi}_l^{(w)} =& \Big(\prod_{(p=w-l+2)}^{w} m^{(p)}(u,\xi_p)\Big)\hat{\Psi}^{(w-l)} \\
\tilde{\Psi}^{(w)} =& n^{(w)}(u,\xi_w)\hat{\Psi}^{(w-2)} \\
\tilde{\Psi}_{k,l}^{(w)} =& \Big(\prod_{(p=w-k+2)}^{w} q^{(p)}(u,\xi_p)\Big)\hat{\Psi}_l^{(w-k)} \\
\chi^{(w)} =& l^{(w)}(u,\xi_w)\tilde{\Gamma}^{(w-2)} \\
\chi_k^{(w)} =& \Big(\prod_{(p=w-k+2)}^{w} n^{(p)}(u,\xi_p)\Big)\chi^{(w-k)} \\
\Omega_{w_b}^{(w)} =& \Big(\prod_{(p=2)}^{w} q^{(p)}(u,\xi_p)\Big)B_\eta(u) \\
\Omega_{w_c}^{(w)} =& \Big(\prod_{(p=2)}^{w} q^{(p)}(u,\xi_p)\Big)C_\eta(u)
\end{aligned}$$

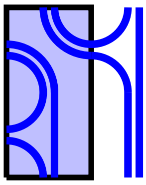
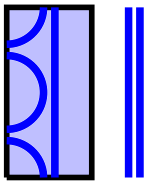
To demonstrate how these expressions are determined we will calculate the coefficient of the following diagram as an example

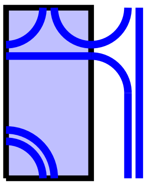
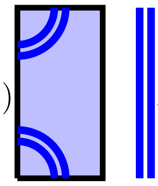


,

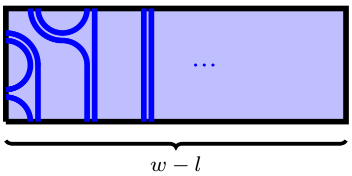
We recognise the key expression defining this object is given by



$= d^{(w)}(u,\xi_w)$  $+\, e^{(w)}(u,\xi_w)$ 

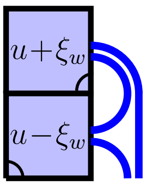
$+\, f^{(w)}(u,\xi_w)$  $+\, g^{(w)}(u,\xi_w)$  .

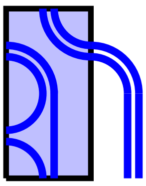
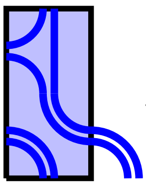
Constructing the intermediate object of width $w-k$ with the corresponding coefficient



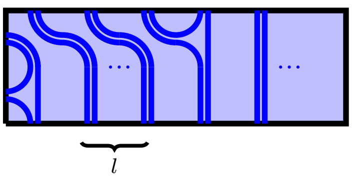
$\rightarrow d^{(w-l)}(u,\xi_{w-l})\Theta^{(w-l-2)}.$

In order to generate the rest of the object we recognise the importance of the expression



$= k^{(w)}(u,\xi_w)$  $+\, l^{(w)}(u,\xi_w)$  . (B.1)

It follows that the application of this result for the remaining $w-k$ blocks, produces the desired operator with the corresponding coefficient



$$\rightarrow \Big(\prod_{p=w-l+2}^{w} k^{(p)}(u,\xi_p)\Big) d^{(w-l)}(u,\xi_{w-l})\Theta^{(w-l-2)} = \Big(\prod_{(p=w-l+2)}^{w} k^{(p)}(u,\xi_p)\Big)\hat{\Gamma}^{(w-l)}$$

Here we have arrived at an expanded form of $\Gamma_l^{(w)}$, as required.

C Matrix Representations

The shear size of the matrices in some contexts are too large to be presented using standard LaTeX formatting. This is true for the Hamiltonian representations and the Gram matrix. For completeness we will present the *Mathematica* code defining these objects for all calculated system sizes.

Throughout the *Mathematica* code blocks we have used the conventions

$$
\begin{aligned}
\rho_N(\mathcal{H}) &= HN \\
\beta &= b \\
\beta_1 &= b1 \\
\beta_2 &= b2 \\
\tilde{\eta} &= n \\
\mathcal{G}_N &= GN
\end{aligned}
$$

C.1 Hamiltonian

$N = 1$

```
H1 = {{-((b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
b2 n)), -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(
b2 n))}, {0, -((b (-b1 + b b2 - b n))/n)}}
```

$N = 2$

```
H2= = {{-2 b (-(1/b) + b), -(b1/b) - (-(1/b) + b) b1, -(b1^2/b) -
b1 (-(b1/b) + b2)}, {-((b (-b1 + b b2 - b n))/(
b2 n)), -b (-(2/b) + b) - (b (-b1 + b b2 - b n))/
n, -b (-((2 b1)/b) + b2)}, {-(b/(b2 n)), -((
b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), -((
b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
b2 n))}}
```

$N = 3$

```
 H3 = {{-b (-(4/b) + 2 b), 0, b1, b1, 0, 0,
0}, {-((b (-b1 + b b2 - b n))/(b2 n)), -b (-(4/b) + 2 b) - (
 b (-b1 + b b2 - b n))/
 n, -b, -b, -b (-(b1/b) + b2), -b (-(b1/b) + b2),
```

```
0}, {-((b (-(b1/b) - (-b1 + b b2 - b n)/b))/(b2 n)), -b,
1 - b^2 - b (-(1/b) + b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
 b2 n), -1, -(b1/b) - (-(1/b) + b) b1, -b2, -(b1^2/b) -
 b1 (-(b1/b) + b2)}, {0, -b, -1,
1 - b^2 -
 b (-(1/b) + b), -b2, -(b1/b) - (-(1/b) + b) b1, -(b1^2/b) -
 b1 (-(b1/b) + b2)}, {-(b/(b2 n)), -((
 b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0,
0, -b (-(2/b) + b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
1, -b (-((2 b1)/b) + b2)}, {0, 0,
0, -((b (-b1 + b b2 - b n))/(b2 n)),
1, -b (-(2/b) + b) - (b (-b1 + b b2 - b n))/
 n, -b (-((2 b1)/b) + b2)}, {0, 0, 0, -(b/(b2 n)),
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), -((
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
 b2 n))}}
```

$N = 4$

```
 H4 = {{-b (-(6/b) + 3 b), -2 b, -b, 0, 0, b1, b1, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0}, {-b, -2 b (-(1/b) + b), -1, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0,
0}, {-2 b, -2, -2 b (-(1/b) + b) - b (-(2/b) + 2 b), 0, 0, 0,
0, -(b1/b) - (-(1/b) + b) b1, -(b1/b) - (-(1/b) + b) b1, 0, 0,
0, -(b1^2/b) - b1 (-(b1/b) + b2), 0, -(b1^2/b) - b1 (-(b1/b) + b2),
0, 0, 0, 0}, {-((b (-b1 + b b2 - b n))/(b2 n)), 0,
0, -b (-(6/b) + 3 b) - (b (-b1 + b b2 - b n))/
 n, -2 b, -b, -b, -b, -b, -b (-(b1/b) + b2), -b b2, -b (-(b1/b) +
   b2), 0, 0, 0, 0, 0, 0, 0}, {0, -((b (-b1 + b b2 - b n))/(b2 n)),
0, -b, -2 b (-(1/b) + b) - (b (-b1 + b b2 - b n))/n, 0, 0, -1, -1,
0, -b2, 0, 0, 0, 0, 0, 0, 0,
0}, {-((b (-(b1/b) - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0,
1, -b (-(4/b) + 2 b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
0, 1, 0, 0, 0, 0, b1, b1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1,
0, -b (-(4/b) + 2 b), 0, 1, 0, 0, 0, 0, b1, b1, 0, 0, 0, 0}, {0,
0, -((b (-b1 + b b2 - b n))/(b2 n)), -b, -1,
1, -1, -b (-(1/b) + b) - b (-(3/b) + 2 b) - (b (-b1 + b b2 - b n))/
 n, 0, 0, -(b1/b) - (-(1/b) + b) b1, -b2, -b (-((2 b1)/b) + b2), 0,
0, 0, 0, -(b1^2/b) - b1 (-(b1/b) + b2), 0}, {0, 0, 0, -b, -1, -1, 1,
 0, -b (-(1/b) + b) -
 b (-(3/b) + 2 b), -b2, -(b1/b) - (-(1/b) + b) b1, 0, 0,
0, -b (-((2 b1)/b) + b2), -(b1^2/b) - b1 (-(b1/b) + b2), 0, 0,
0}, {-(b/(b2 n)), 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0,
0, -b (-(4/b) + 2 b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
1, 0, -b, -b, 0, -b (-(b1/b) + b2), -b (-(b1/b) + b2), 0, 0}, {0, 0,
 0, 0, 0, 0, 0, 0, -((b (-b1 + b b2 - b n))/(b2 n)),
1, -b (-(4/b) + 2 b) - (b (-b1 + b b2 - b n))/n, 1, 0, 0,
0, -b (-((2 b1)/b) + b2), 0, -b (-((2 b1)/b) + b2), 0}, {0, 0, 0, 0,
 0, 0, -((b (-b1 + b b2 - b n))/(b2 n)), 0, 0, 0,
1, -b (-(4/b) + 2 b) - (b (-b1 + b b2 - b n))/n, 0, -b, -b,
0, -b (-(b1/b) + b2), -b (-(b1/b) + b2), 0}, {0, 0, -(b/(b2 n)), 0,
0, 0, 0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, -b,
 0, 0, -2 b (-(1/b) + b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
 b2 n), -1, 0, -(b1/b) - (-(1/b) + b) b1, -b2,
0, -(b1^2/b) - b1 (-(b1/b) + b2)}, {0, -(b/(b2 n)), 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)),
0, -((b (-(b1/b) - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, -b,
0, -b, -1, -2 b (-(1/b) + b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
 b2 n), -1, -b2, -(b1/b) - (-(1/b) + b) b1, -b2, -(b1^2/b) -
 b1 (-(b1/b) + b2)}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -b,
0, -1, -2 b (-(1/b) + b),
0, -b2, -(b1/b) - (-(1/b) + b) b1, -(b1^2/b) -
 b1 (-(b1/b) + b2)}, {0, 0, 0, 0, 0, 0, 0, 0, -(b/(b2 n)),
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0,
0, -b (-(2/b) + b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
1, 0, -b (-((2 b1)/b) + b2)}, {0, 0, 0, 0, 0, 0, -(b/(b2 n)), 0, 0,
0, 0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0,
1, -b (-(2/b) + b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
1, -b (-((2 b1)/b) + b2)}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -((b (-b1 + b b2 - b n))/(b2 n)), 0,
1, -b (-(2/b) + b) - (b (-b1 + b b2 - b n))/
 n, -b (-((2 b1)/b) + b2)}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -(b/(b2 n)), 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), -((
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
 b2 n))}}
```

$N = 5$

```
 H5 = {{-b (-(8/b) + 4 b), -2 b, -b, -2 b, -2 b, -b, 0, b1, b1, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0}, {0, -b (-(4/b) + 2 b), 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0}, {-b, -1, -b (-(1/b) + b) - b (-(5/b) + 3 b), -1, 1, 1, 0, 0, 0,
0, 0, -(b1/b) - (-(1/b) + b) b1, 0, 0, 0, 0, b1, b1, 0, 0, 0, 0, 0,
0, 0, 0, 0, -(b1/b) - b1 (-(b1/b) + b2), 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {-b,
1, -1, -b (-(1/b) + b) - b (-(3/b) + 2 b), -1, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, b1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {-b, 1,
0, -1, -b (-(1/b) + b) - b (-(3/b) + 2 b), -1, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, b1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {-b, -1, 1,
1, -1, -b (-(1/b) + b) - b (-(5/b) + 3 b), 0, 0,
0, -(b1/b) - (-(1/b) + b) b1, 0, 0, 0, 0, 0, 0, 0, b1, b1, 0, 0, 0,
0, 0, 0, -(b1^2/b) - b1 (-(b1/b) + b2), 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0}, {-((b (-b1 + b b2 - b n))/(b2 n)), 0, 0, 0, 0,
0, -b (-(8/b) + 4 b) - (b (-b1 + b b2 - b n))/
 n, -b, -b, -b, -2 b, -b, -b, -2 b, -2 b, -b, 0, 0, 0, 0,
0, -b (-(b1/b) + b2), -b b2, -b b2, -b (-(b1/b) + b2), 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0}, {-((b (-(b1/b) - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0, 0,
0, -b (-(6/b) + 3 b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
0, 1, 1, 0, 0, 0, 0, 0, -b, 0, 0, -2 b, 0, 0, 0, 0, 0, b1, b1, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0}, {0, 0, 0, 0, 0, 0, 0, 0, -b (-(6/b) + 3 b), 0, 1, 1, 0, 0, 0, 0,
 0, 0, -b, 0, -2 b, 0, 0, 0, 0, 0, b1, b1, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0,
0, -((b (-b1 + b b2 - b n))/(b2 n)), 0, 1,
0, -b (-(6/b) + 3 b) - (b (-b1 + b b2 - b n))/n, 0, 0, 1, 1, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, -b (-((2 b1)/b) + b2), 0, 0, b1, b1, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0}, {0, -((b (-b1 + b b2 - b n))/(b2 n)), 0, 0, 0, 0, 0, 0, 0,
0, -b (-(4/b) + 2 b) - (b (-b1 + b b2 - b n))/n, 0, 0, 1, 1, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 1, 0,
0, -b (-(6/b) + 3 b), 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -b (-((2 b1)/b) + b2), 0, 0, 0, 0, 0, 0, b1, b1, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0,
0, -((b (-b1 + b b2 - b n))/(b2 n)), 0, 0, 0, -b, 0, -1, 1, -1,
0, -b (-(1/b) + b) - b (-(5/b) + 3 b) - (b (-b1 + b b2 - b n))/
 n, -1, 0, 0, -b, -b, 0, 0, 0, 0, 0, -(b1/b) - (-(1/b) + b) b1, -b2,
 0, 0, 0, -b (-(b1/b) + b2), 0, 0, 0, -b (-(b1/b) + b2), 0, 0, 0, 0,
 0, -(b1^2/b) - b1 (-(b1/b) + b2), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0}, {0, 0, 0, -((b (-b1 + b b2 - b n))/(b2 n)), 0, 0, -b, 0, 0, 0,
1, -1, -1, -b (-(1/b) + b) - b (-(3/b) + 2 b) - (
 b (-b1 + b b2 - b n))/n, -1, 0, 0, 0, 0, -b, 0, 0, 0, -b2, 0, 0, 0,
 0, 0, 0, 0, 0, 0, -b (-(b1/b) + b2), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, -((b (-b1 + b b2 - b n))/(b2 n)),
 0, -b, 0, 0, -1, 1, 0,
0, -1, -b (-(1/b) + b) - b (-(3/b) + 2 b) - (b (-b1 + b b2 - b n))/
 n, -1, 0, 0, 0, 0, -b, 0, -b2, 0, 0, 0, 0, 0, 0,
0, -b (-(b1/b) + b2), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -b, -1, 0, 0, -1, 1, 0,
0, -1, -b (-(1/b) + b) - b (-(5/b) + 3 b), 0, -b, -b, 0,
0, -b2, -(b1/b) - (-(1/b) + b) b1, 0, 0, 0, 0, 0, 0, 0,
0, -b (-(b1/b) + b2), 0, 0,
0, -b (-(b1/b) + b2), -(b1^2/b) - b1 (-(b1/b) + b2), 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0}, {0,
0, -((b (-(b1/b) - (-b1 + b b2 - b n)/b))/(b2 n)),
0, -((b (-(b1/b) - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, -2 b, 0, 0,
 0, 0, -b, 0, -b,
0, -2 b (-(1/b) + b) - b (-(2/b) + 2 b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
 b2 n), -1, 0, -2, -1, 0, 0, 0, 0, 0, 0, 0, -b2, 0, -b2,
0, -(b1/b) - (-(1/b) + b) b1, 0, -(b1/b) - (-(1/b) + b) b1, 0, 0, 0,
 0, 0, 0, 0, -(b1^2/b) - b1 (-(b1/b) + b2),
0, -(b1^2/b) - b1 (-(b1/b) + b2), 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, -b, 0,
0, -b, -1, -2 b (-(1/b) + b) - b (-(2/b) + 2 b), -1, 0, 0, 0, 0, 0,
0, 0, 0, 0, -(b1/b) - (-(1/b) + b) b1, 0, 0, -b2, -b2, 0,
0, -(b1/b) - (-(1/b) + b) b1, 0, 0, 0, 0, 0,
0, -(b1^2/b) - b1 (-(b1/b) + b2), 0,
0, -(b1^2/b) - b1 (-(b1/b) + b2), 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0,
 0, 0, -2 b, 0, 0, 0, 0, -b, 0, -b,
0, -1, -2 b (-(1/b) + b) - b (-(2/b) + 2 b), -1, -2, 0, 0, 0, 0, 0,
0, 0, 0, -(b1/b) - (-(1/b) + b) b1, 0, -(b1/b) - (-(1/b) + b) b1,
0, -b2, 0, -b2, 0, 0, 0, 0, 0, 0, 0, -(b1^2/b) - b1 (-(b1/b) + b2),
0, -(b1^2/b) - b1 (-(b1/b) + b2), 0, 0, 0, 0, 0}, {0, 0,
0, -((b (-(b1/b) - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, -b, 0,
0, 0, 0, 0, 0, 0, 0, -1, 0,
0, -2 b (-(1/b) + b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
```

```
 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, -b, 0, 0, 0, 0,
0, 0, 0, 0, 0, -1, 0, 0, -2 b (-(1/b) + b), 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0}, {-(b/(b2 n)), 0, 0, 0, 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -b (-(6/b) + 3 b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
1, 0, 0, -b, -b, 0, 0, 0, 0, 0, -b, -2 b, -b,
0, -b (-(b1/b) + b2), -b (-(b1/b) + b2), 0, -b b2, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -((b (-b1 + b b2 - b n))/(b2 n)), 0, 0, 0, 0, 0, 0,
1, -b (-(6/b) + 3 b) - (b (-b1 + b b2 - b n))/n, 1, 0, 0, 0,
0, -b, -b, 0, 0, 0, 0, 0, 0, -b (-((2 b1)/b) + b2), 0, 0,
0, -b (-(b1/b) + b2), -b (-(b1/b) + b2), 0, 0, 0, 0, 0, 0, 0, 0, 0,
0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -((b (-b1 + b b2 - b n))/(b2 n)), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
1, -b (-(6/b) + 3 b) - (b (-b1 + b b2 - b n))/n, 1, 0, 0, 0, 0, 0,
0, 0, 0, 0, -b, -b, 0,
0, -b (-((2 b1)/b) + b2), -b (-(b1/b) + b2), -b (-(b1/b) + b2), 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0,
0, -((b (-b1 + b b2 - b n))/(b2 n)), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 1, -b (-(6/b) + 3 b) - (b (-b1 + b b2 - b n))/n,
0, -b, -b, 0, -b, -2 b, -b, 0, 0, 0, 0,
0, -b (-(b1/b) + b2), -b (-(b1/b) + b2), 0, 0, -b b2, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -(b/(b2 n)), 0, 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -b (-(4/b) + 2 b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, b1, b1, 0, 0, 0, 0,
0, 0, 0}, {0, -(b/(b2 n)), 0, 0, 0, 0, 0, 0,
0, -((b (-(b1/b) - (-b1 + b b2 - b n)/b))/(b2 n)),
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -b (-(4/b) + 2 b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, b1, b1, 0, 0, 0, 0,
0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, -b (-(4/b) + 2 b), 0, 0, 1, 1, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, b1, b1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -((b (-b1 + b b2 - b n))/(b2 n)), 0,
0, 0, 0, -b, 0, 0, 0, 0, 0,
0, -b (-(1/b) + b) - b (-(3/b) + 2 b) - (b (-b1 + b b2 - b n))/
 n, -1, 0, 0, 1, 0, 0, 0, 0, 0, 0,
0, -(b1/b) - (-(1/b) + b) b1, -b2, -b (-((2 b1)/b) + b2), 0, 0, 0,
0, 0, 0, -(b1^2/b) - b1 (-(b1/b) + b2), 0}, {0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -((b (-b1 + b b2 - b n))/(b2 n)), 0,
0, 0, -b, 0, -b, 0,
1, -1, -1, -b (-(1/b) + b) - b (-(3/b) + 2 b) - (
 b (-b1 + b b2 - b n))/n, -1, 0, 0, 1, 0, 0, 0, 0, -b2,
0, -b2, -(b1/b) - (-(1/b) + b) b1, 0, -b (-((2 b1)/b) + b2), 0, 0,
0, 0, 0, -(b1^2/b) - b1 (-(b1/b) + b2), 0}, {0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -((b (-b1 + b b2 - b n))/(b2 n)), 0, 0, 0, -b, 0, 0, 0,
0, -1, -2 b (-(1/b) + b) - (b (-b1 + b b2 - b n))/n, -1, 0, 0, 0, 0,
 0, 0, 0, 0, 0, -b2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -b, 0, -1, 1,
0, 0, -1, -b (-(1/b) + b) - b (-(3/b) + 2 b), 0, 0, 0, 1, 0, -b2, 0,
 0, 0, -(b1/b) - (-(1/b) + b) b1, 0, 0, 0, -b (-((2 b1)/b) + b2),
0, -(b1^2/b) - b1 (-(b1/b) + b2), 0, 0, 0}, {0, 0, -(b/(b2 n)), 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0,
0, 0, 0, 0, -b, 0, 0, 0, 1, -1, 0, 1, 0, 0,
0, -b (-(1/b) + b) - b (-(3/b) + 2 b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
 b2 n), -1, 0, 0, 0, -b2, 0, -(b1/b) - (-(1/b) + b) b1, 0,
0, -b (-((2 b1)/b) + b2), 0, 0, 0, 0,
0, -(b1^2/b) - b1 (-(b1/b) + b2), 0, 0}, {0, 0, 0, -(b/(b2 n)), 0,
0, 0, 0, 0, 0, 0, 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0,
0, 0, 0, -b, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -1, -2 b (-(1/b) + b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
 b2 n), -1, 0, 0, 0, 0, -b2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0,
0, 0, 0, -(b/(b2 n)), 0, 0, 0, 0, 0,
0, -((b (-(b1/b) - (-b1 + b b2 - b n)/b))/(b2 n)), 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0,
0, 0, -b, 0, -b, 0, -1, 1, 0, 0, 0, 1, 0,
0, -1, -b (-(1/b) + b) - b (-(3/b) + 2 b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
 b2 n), -1, -b2, 0, 0, -(b1/b) - (-(1/b) + b) b1, -b2, 0, 0,
0, -b (-((2 b1)/b) + b2), 0, -(b1^2/b) - b1 (-(b1/b) + b2), 0, 0, 0,
 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, -b, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0,
0, -1, -b (-(1/b) + b) - b (-(3/b) + 2 b), 0, 0,
0, -b2, -(b1/b) - (-(1/b) + b) b1, 0, 0, 0, 0,
0, -b (-((2 b1)/b) + b2), -(b1^2/b) - b1 (-(b1/b) + b2), 0, 0, 0,
0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -(b/(b2 n)), 0, 0,
 0, 0, 0, 0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -b (-(4/b) + 2 b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
0, 0, 1, 0, 0, -b, -b, 0, 0, -b (-(b1/b) + b2), -b (-(b1/b) + b2),
0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, -(b/(b2 n)), 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0,
0, -b (-(4/b) + 2 b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
0, 1, 0, 1, 0, -b, -b, 0, 0, -b (-(b1/b) + b2), -b (-(b1/b) + b2),
0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, -((b (-b1 + b b2 - b n))/(b2 n)), 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, -b (-(4/b) + 2 b) - (b (-b1 + b b2 - b n))/n, 0,
0, 1, 0, 0, -b, -b, 0, 0, -b (-(b1/b) + b2), -b (-(b1/b) + b2),
0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -(b/(b2 n)), 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)),
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1,
0, -b (-(4/b) + 2 b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
1, 0, 0, 0, 0, 0, -b (-((2 b1)/b) + b2), 0, -b (-((2 b1)/b) + b2),
0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -((b (-b1 + b b2 - b n))/(b2 n)), 0, 0, 0,
1, -b (-(4/b) + 2 b) - (b (-b1 + b b2 - b n))/n, 1, 0, 0, 0,
0, -b (-((2 b1)/b) + b2), 0, 0, -b (-((2 b1)/b) + b2), 0}, {0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, -((b (-b1 + b b2 - b n))/(b2 n)), 0, 0, 0, 0, 0,
1, 1, 0, 1, -b (-(4/b) + 2 b) - (b (-b1 + b b2 - b n))/n, 0, 0, 0,
0, 0, -b (-((2 b1)/b) + b2), 0, -b (-((2 b1)/b) + b2), 0}, {0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -(b/(b2 n)), 0, 0, 0, 0,
0, 0, 0, 0, 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0,
0, 0, 0, -b, 0, 0, 0, 0,
0, -2 b (-(1/b) + b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
 b2 n), -1, 0, 0, -(b1/b) - (-(1/b) + b) b1, -b2, 0,
0, -(b1^2/b) - b1 (-(b1/b) + b2)}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, -(b/(b2 n)), 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0,
0, 0, -b, -b, 0, 0, 0,
0, -1, -2 b (-(1/b) + b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
 b2 n), -1, 0, -b2, -(b1/b) - (-(1/b) + b) b1, -b2,
0, -(b1^2/b) - b1 (-(b1/b) + b2)}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, -(b/(b2 n)), 0, 0, 0, 0, 0,
0, -((b (-(b1/b) - (-b1 + b b2 - b n)/b))/(b2 n)), 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0,
0, 0, -b, -b, 0, 0, 0,
0, -1, -2 b (-(1/b) + b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
 b2 n), -1,
0, -b2, -(b1/b) - (-(1/b) + b) b1, -b2, -(b1^2/b) -
 b1 (-(b1/b) + b2)}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -b, 0, 0, 0, 0, 0, -1, -2 b (-(1/b) + b), 0,
0, -b2, -(b1/b) - (-(1/b) + b) b1, -(b1^2/b) - b1 (-(b1/b) + b2)},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -(b/(b2 n)), 0, 0, 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0,
0, -b (-(2/b) + b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
1, 0, 0, -b (-((2 b1)/b) + b2)}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -(b/(b2 n)), 0, 0, 0, 0, 0, 0, 0, 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0,
1, -b (-(2/b) + b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
1, 0, -b (-((2 b1)/b) + b2)}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -(b/(b2 n)), 0, 0, 0,
0, 0, 0, 0, 0, 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), 0, 0, 0, 0,
0, 0, 0, 0,
1, -b (-(2/b) + b) - (
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(b2 n),
1, -b (-((2 b1)/b) + b2)}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, -b (-(b1/b) + b2), 0, 0, 0,
0, -((b (-b1 + b b2 - b n))/(b2 n)), 0, 0,
1, -b (-(2/b) + b) - (b (-b1 + b b2 - b n))/
 n, -b (-((2 b1)/b) + b2)}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, -(b/(b2 n)), 0, 0,
0, -((b (-(b1/b) + b2 - (-b1 + b b2 - b n)/b))/(b2 n)), -((
 b (b1 (-(b1/b) + b2) + (-(b1/b) + b2) (-b1 + b b2 - b n)))/(
 b2 n))}}
```

C.2 Gram Matrices

N=4

```
 G4 = {{b (-(1/b) + b) (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b)^2, b (-(1/b) + b)^2}, {b (-(1/b) + b)^2,
b^2 (-(1/b) + b)^2, b (-(1/b) + b)}, {b (-(1/b) + b)^2,
b (-(1/b) + b), b^2 (-(1/b) + b)^2}}
```

N=5

```
 G5 = {{b (-(2/b) + b) (-(1/b) + b) (1 + (-(2/b) + b) (-(1/b) + b)),
b (-(2/b) + b) (-(1/b) + b)^2, b (-(2/b) + b) (-(1/b) + b)^2,
b (-(2/b) + b) (-(1/b) + b)^2, b (-(2/b) + b) (-(1/b) + b)^2,
b (-(2/b) + b) (-(1/b) + b)^2}, {b (-(2/b) + b) (-(1/b) + b)^2,
b^2 (-(2/b) + b) (-(1/b) + b)^2, b (-(2/b) + b) (-(1/b) + b), 0, 0,
b (-(2/b) + b) (-(1/b) + b)}, {b (-(2/b) + b) (-(1/b) + b)^2,
b (-(2/b) + b) (-(1/b) + b), b^2 (-(2/b) + b) (-(1/b) + b)^2,
b (-(2/b) + b) (-(1/b) + b), 0, 0}, {b (-(2/b) + b) (-(1/b) + b)^2,
0, b (-(2/b) + b) (-(1/b) + b), b^2 (-(2/b) + b) (-(1/b) + b)^2,
b (-(2/b) + b) (-(1/b) + b), 0}, {b (-(2/b) + b) (-(1/b) + b)^2, 0,
0, b (-(2/b) + b) (-(1/b) + b), b^2 (-(2/b) + b) (-(1/b) + b)^2,
b (-(2/b) + b) (-(1/b) + b)}, {b (-(2/b) + b) (-(1/b) + b)^2,
b (-(2/b) + b) (-(1/b) + b), 0, 0, b (-(2/b) + b) (-(1/b) + b),
b^2 (-(2/b) + b) (-(1/b) + b)^2}}
```

N=6

```
 G6 = {{b^2 (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b) (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b)^3, b (-(1/b) + b)^3, b (-(1/b) + b)^3,
b (-(1/b) + b) (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(2/b) + b)^2 (-(1/b) + b), 0, 0, b^2 (-(1/b) + b)^3,
b^2 (-(1/b) + b)^3, b (-(1/b) + b)^2, b (-(1/b) + b)^2,
b (-(1/b) + b)^2,
b (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b))}, {b (-(1/b) +
   b) (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b^2 (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b) (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b)^3, b (-(1/b) + b)^3, b (-(1/b) + b)^3, 0,
b (-(2/b) + b)^2 (-(1/b) + b), 0, b (-(1/b) + b)^2,
b (-(1/b) + b)^2, b^2 (-(1/b) + b)^3, b (-(1/b) + b)^2,
b^2 (-(1/b) + b)^3,
b (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b))}, {b (-(1/b) +
   b)^3, b (-(1/b) + b) (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b^2 (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b) (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b)^3, b (-(1/b) + b)^3, 0, 0,
b (-(2/b) + b)^2 (-(1/b) + b), b^2 (-(1/b) + b)^3, b (-(1/b) + b)^2,
 b (-(1/b) + b)^2, b^2 (-(1/b) + b)^3, b (-(1/b) + b)^2,
b (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b))}, {b (-(1/b) +
   b)^3, b (-(1/b) + b)^3,
b (-(1/b) + b) (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b^2 (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b) (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b)^3, b (-(2/b) + b)^2 (-(1/b) + b), 0, 0,
b (-(1/b) + b)^2, b^2 (-(1/b) + b)^3, b^2 (-(1/b) + b)^3,
b (-(1/b) + b)^2, b (-(1/b) + b)^2,
b (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b))}, {b (-(1/b) +
   b)^3, b (-(1/b) + b)^3, b (-(1/b) + b)^3,
b (-(1/b) + b) (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b^2 (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b) (1/b^2 + (-(2/b) + b) (-(1/b) + b)), 0,
b (-(2/b) + b)^2 (-(1/b) + b), 0, b^2 (-(1/b) + b)^3,
b (-(1/b) + b)^2, b (-(1/b) + b)^2, b (-(1/b) + b)^2,
b^2 (-(1/b) + b)^3,
b (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b))}, {b (-(1/b) +
   b) (1/b^2 + (-(2/b) + b) (-(1/b) + b)), b (-(1/b) + b)^3,
b (-(1/b) + b)^3, b (-(1/b) + b)^3,
b (-(1/b) + b) (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b^2 (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b)), 0, 0,
b (-(2/b) + b)^2 (-(1/b) + b), b (-(1/b) + b)^2, b (-(1/b) + b)^2,
b^2 (-(1/b) + b)^3, b^2 (-(1/b) + b)^3, b (-(1/b) + b)^2,
b (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b))}, {b (-(2/b) +
   b)^2 (-(1/b) + b), 0, 0, b (-(2/b) + b)^2 (-(1/b) + b), 0, 0,
b^2 (-(2/b) + b)^2 (-(1/b) + b)^2, b (-(2/b) + b)^2 (-(1/b) + b),
b (-(2/b) + b)^2 (-(1/b) + b), 0, b (-(2/b) + b) (-(1/b) + b), 0, 0,
 0, b (-(2/b) + b)^2 (-(1/b) + b)^2}, {0,
b (-(2/b) + b)^2 (-(1/b) + b), 0, 0, b (-(2/b) + b)^2 (-(1/b) + b),
0, b (-(2/b) + b)^2 (-(1/b) + b), b^2 (-(2/b) + b)^2 (-(1/b) + b)^2,
 b (-(2/b) + b)^2 (-(1/b) + b), 0, 0, 0, 0,
b (-(2/b) + b) (-(1/b) + b), b (-(2/b) + b)^2 (-(1/b) + b)^2}, {0,
0, b (-(2/b) + b)^2 (-(1/b) + b), 0, 0,
b (-(2/b) + b)^2 (-(1/b) + b), b (-(2/b) + b)^2 (-(1/b) + b),
b (-(2/b) + b)^2 (-(1/b) + b), b^2 (-(2/b) + b)^2 (-(1/b) + b)^2, 0,
 0, 0, b (-(2/b) + b) (-(1/b) + b), 0,
b (-(2/b) + b)^2 (-(1/b) + b)^2}, {b^2 (-(1/b) + b)^3,
b (-(1/b) + b)^2, b^2 (-(1/b) + b)^3, b (-(1/b) + b)^2,
b^2 (-(1/b) + b)^3, b (-(1/b) + b)^2, 0, 0, 0, b^3 (-(1/b) + b)^3,
b^2 (-(1/b) + b)^2, b (-(1/b) + b), b^2 (-(1/b) + b)^2,
b^2 (-(1/b) + b)^2, b (-(1/b) + b)^3}, {b^2 (-(1/b) + b)^3,
b (-(1/b) + b)^2, b (-(1/b) + b)^2, b^2 (-(1/b) + b)^3,
b (-(1/b) + b)^2, b (-(1/b) + b)^2, b (-(2/b) + b) (-(1/b) + b), 0,
0, b^2 (-(1/b) + b)^2, b^3 (-(1/b) + b)^3, b^2 (-(1/b) + b)^2,
b (-(1/b) + b), b (-(1/b) + b),
b (-(1/b) + b)^3}, {b (-(1/b) + b)^2, b^2 (-(1/b) + b)^3,
b (-(1/b) + b)^2, b^2 (-(1/b) + b)^3, b (-(1/b) + b)^2,
b^2 (-(1/b) + b)^3, 0, 0, 0, b (-(1/b) + b), b^2 (-(1/b) + b)^2,
b^3 (-(1/b) + b)^3, b^2 (-(1/b) + b)^2, b^2 (-(1/b) + b)^2,
b (-(1/b) + b)^3}, {b (-(1/b) + b)^2, b (-(1/b) + b)^2,
b^2 (-(1/b) + b)^3, b (-(1/b) + b)^2, b (-(1/b) + b)^2,
b^2 (-(1/b) + b)^3, 0, 0, b (-(2/b) + b) (-(1/b) + b),
b^2 (-(1/b) + b)^2, b (-(1/b) + b), b^2 (-(1/b) + b)^2,
b^3 (-(1/b) + b)^3, b (-(1/b) + b),
b (-(1/b) + b)^3}, {b (-(1/b) + b)^2, b^2 (-(1/b) + b)^3,
b (-(1/b) + b)^2, b (-(1/b) + b)^2, b^2 (-(1/b) + b)^3,
b (-(1/b) + b)^2, 0, b (-(2/b) + b) (-(1/b) + b), 0,
b^2 (-(1/b) + b)^2, b (-(1/b) + b), b^2 (-(1/b) + b)^2,
b (-(1/b) + b), b^3 (-(1/b) + b)^3,
b (-(1/b) + b)^3}, {b (-(1/b) + b)^2 (1/
   b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(1/b) + b)^2 (1/b^2 + (-(2/b) + b) (-(1/b) + b)),
b (-(2/b) + b)^2 (-(1/b) + b)^2, b (-(2/b) + b)^2 (-(1/b) + b)^2,
b (-(2/b) + b)^2 (-(1/b) + b)^2, b (-(1/b) + b)^3, b (-(1/b) + b)^3,
 b (-(1/b) + b)^3, b (-(1/b) + b)^3, b (-(1/b) + b)^3,
b (-(1/b) + b) (b^4 - (5 (-(1/b) + b) (1 + (-(1/b) + b)^2))/b)}}
```

N=7

```
 G7 = {{((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (2 - 2 b^2 + b^4))/b^3,
0, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/b^3,
0, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^4}, {((-1 + b) (1 + b) (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/b^3, 0,
0, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
```

```
 b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^4}, {((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3, 0,
0, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^4}, {((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3, 0,
0, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^4}, {((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3, 0,
0, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^4}, {((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3,
0, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^4}, {((-1 + b) (1 + b) (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/b^4}, {0,
0, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
 b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2)^3)/b^3, 0,
0, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
 b^3, ((-1 + b) (1 + b) (-2 + b^2)^3)/b^3, 0, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/
```

```
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^4}, {((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3, 0,
0, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b) (1 + b) (-2 + b^2)^3)/b^3,
0, ((-1 + b) (1 + b) (-2 + b^2)^3)/b^3, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^4}, {((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3, 0,
0, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b) (1 + b) (-2 + b^2)^3)/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b) (1 + b) (-2 + b^2)^3)/b^3, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^4}, {((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3, 0,
0, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/b^3,
0, ((-1 + b) (1 + b) (-2 + b^2)^3)/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3,
0, ((-1 + b) (1 + b) (-2 + b^2)^3)/
b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^4}, {((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3, 0,
0, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^3)/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b) (1 + b) (-2 + b^2)^3)/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0, 0, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^4}, {0, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3,
0, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b) (1 + b) (-2 + b^2)^3)/b^3, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^3)/
b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^4}, {((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/b^3, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^3)/b^3, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^3)/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^3, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^4}, {((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0, 0, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3}, {((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b, ((-1 + b) (1 + b) (-2 + b^2))/b, ((-1 + b) (1 + b) (-2 + b^2))/
b, 0, 0, ((-1 + b) (1 + b) (-2 + b^2))/
b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0, 0,
0, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3}, {((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
```

```
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/b, ((-1 + b) (1 + b) (-2 + b^2))/b,
 0, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/b, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0, 0, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/b, ((-1 + b) (1 + b) (-2 + b^2))/b,
 0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0,
0, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/b^3}, {0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b,
0, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0, 0,
0, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0, 0, 0, 0, 0, 0,
0, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b,
0, ((-1 + b) (1 + b) (-2 + b^2))/b, ((-1 + b) (1 + b) (-2 + b^2))/b,
 0, 0, 0, 0, 0, 0, 0,
0, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0, 0, 0, 0, 0, 0, 0,
0, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0, 0, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, ((-1 + b) (1 + b) (-2 + b^2))/
 b, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, ((-1 + b) (1 + b) (-2 + b^2))/
 b, 0, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
```

```
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/b,
0, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/b^3}, {0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/b, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/b, ((-1 + b) (1 + b) (-2 + b^2))/b,
 0, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b,
0, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/b,
0, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/b, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0, 0, 0, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/b, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/b^2, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/b, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b^2, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b^2,
0, ((-1 + b) (1 + b) (-2 + b^2)^2)/
 b^2, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/b, 0,
0, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, 0, 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/b, ((-1 + b) (1 + b) (-2 + b^2))/b,
 0, 0, 0,
0, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^2 (1 + b)^2 (-2 + b^2))/
 b, ((-1 + b) (1 + b) (-2 + b^2))/b, ((-1 + b) (1 + b) (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
 b^3}, {((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^4, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^4, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^4, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^4, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^4, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^4, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (2 - 2 b^2 + b^4))/
 b^4, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
```

```
b^4, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^4, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^4, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^4, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^4, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^4, ((-1 + b)^2 (1 + b)^2 (-2 + b^2) (3 - 3 b^2 + b^4))/
b^4, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b)^3 (1 + b)^3 (-2 + b^2))/
b^3, ((-1 + b) (1 + b) (-2 + b^2) (3 - 3 b^2 + b^4) (1 - b^2 +
    b^4))/b^5}}
```

C.3 Eigenvectors

The eigenvectors of the matrix representations of the Hamiltonian for $N = 2, 3, 4, 5$ are as follows

$$v^{(2)} = \begin{pmatrix} \frac{2\sqrt{3}}{\sqrt{3}-3} \\ -\frac{2\sqrt{3}}{\sqrt{3}-3} \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{2\sqrt{3}}{\sqrt{3}+3} \\ -\frac{2\sqrt{3}}{\sqrt{3}+3} \\ 1 \end{pmatrix}$$

$$v^{(3)} = \begin{pmatrix} -35.1154 \\ 35.1154 \\ -21.6170 \\ -12.7427 \\ 14.1055 \\ 9.34981 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 3.72816 \\ -3.72816 \\ 0.794531 \\ -1.51966 \\ 2.11204 \\ 2.56533 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0.542583 \\ -0.542583 \\ -1.06070 \\ -0.961199 \\ 0.189673 \\ -1.24584 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1.00000 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ -2.00000 \\ 0 \\ 0 \\ 0 \\ 1.00000 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0.844682 \\ -0.844682 \\ -4.11681 \\ 3.22353 \\ 1.59274 \\ -0.669300 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1.00000 \\ 0 \\ 1.00000 \end{pmatrix}$$

$$v^{(4)} = \begin{pmatrix} -2060.26 \\ 164.087 \\ 1307.80 \\ 2060.26 \\ -164.087 \\ -1114.49 \\ -516.606 \\ -1111.98 \\ -803.483 \\ 855.990 \\ 607.658 \\ 416.234 \\ -248.393 \\ -327.195 \\ -41.0523 \\ 78.0927 \\ 73.1047 \\ 25.1119 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 114.407 \\ -19.9919 \\ -40.8118 \\ -114.407 \\ 19.9919 \\ 33.2197 \\ -36.0767 \\ 21.2945 \\ -33.4180 \\ 22.8233 \\ 52.9353 \\ 50.8121 \\ -10.9403 \\ -31.6994 \\ -15.5281 \\ 23.4336 \\ 25.6048 \\ 11.4453 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4.87897 \\ 7.20147 \\ 0 \\ 0 \\ 4.72545 \\ 3.04795 \\ -3.84648 \\ 3.20147 \\ -3.20147 \\ -4.72545 \\ -7.20147 \\ -3.04795 \\ -1.67750 \\ 1.32250 \\ -1.67750 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 13.1441 \\ -6.54987 \\ 6.08091 \\ -13.1441 \\ 6.54987 \\ -29.3688 \\ 21.4980 \\ -23.1186 \\ 13.7091 \\ 15.2110 \\ 3.32862 \\ -9.30919 \\ -19.4708 \\ 2.00234 \\ -2.15675 \\ 10.2352 \\ 8.62377 \\ 4.01356 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -8.63824 \\ 1.25343 \\ 0 \\ 0 \\ 0.398534 \\ 6.29021 \\ -13.0368 \\ -2.74657 \\ 2.74657 \\ -0.398534 \\ -1.25343 \\ -6.29021 \\ 5.89167 \\ 8.89167 \\ 5.89167 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} -1.53400 \\ -1.94687 \\ -3.41948 \\ 1.53400 \\ 1.94687 \\ 0.752844 \\ 0.438931 \\ -0.247683 \\ 1.39743 \\ -1.62622 \\ 2.26973 \\ -2.02288 \\ -0.951341 \\ -3.40202 \\ -1.93693 \\ -1.42228 \\ 1.07257 \\ -1.57586 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4.93923 \\ -10.5176 \\ -4.93923 \\ 0 \\ 0 \\ 0 \\ -4.93923 \\ 3.25879 \\ 6.25879 \\ 4.93923 \\ 1.00000 \end{pmatrix},$$

$$\begin{pmatrix} 3.02569 \\ 17.3836 \\ -42.6113 \\ -3.02569 \\ -17.3836 \\ -12.0744 \\ 2.04336 \\ 26.6996 \\ 25.5698 \\ 10.8401 \\ -9.65812 \\ -3.19588 \\ -1.30522 \\ -12.8703 \\ 5.25057 \\ 5.93861 \\ 3.45786 \\ -0.348107 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 1.49049 \\ 3.52377 \\ 3.52509 \\ -1.49049 \\ -3.52377 \\ 1.70643 \\ 2.27323 \\ -0.793172 \\ -1.94829 \\ -0.216501 \\ -0.783627 \\ 0.308188 \\ -1.11481 \\ -1.15522 \\ -1.15479 \\ 0.704994 \\ -0.950678 \\ -0.845964 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1.41421 \\ 0 \\ 0 \\ 1.41421 \\ -0.585786 \\ 1.41421 \\ 0 \\ 0 \\ -1.41421 \\ 1.00000 \\ -1.41421 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.41421 \\ 0 \\ 0 \\ -1.41421 \\ -3.41421 \\ -1.41421 \\ 0 \\ 0 \\ 1.41421 \\ 1.00000 \\ 1.41421 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0.130683 \\ -0.198932 \\ 0.138191 \\ -0.130683 \\ 0.198932 \\ -0.227195 \\ -0.247897 \\ 1.27816 \\ -0.0612406 \\ 0.0390715 \\ -1.35511 \\ -3.21470 \\ -1.34268 \\ -0.0524352 \\ 0.0754827 \\ 1.32410 \\ 0.851268 \\ 1.31385 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2.40673 \\ 3.02112 \\ -2.40673 \\ 0 \\ 0 \\ 0 \\ -2.40673 \\ -3.51056 \\ -0.510558 \\ 2.40673 \\ 1.00000 \end{pmatrix},$$

$$\begin{pmatrix} -2.40018 \\ -4.30649 \\ 7.29555 \\ 2.40018 \\ 4.30649 \\ 2.47723 \\ 0.677509 \\ -3.13484 \\ -1.76600 \\ -1.06042 \\ -2.39471 \\ 0.388564 \\ 5.51817 \\ -7.62821 \\ 4.50285 \\ -0.306915 \\ 2.23574 \\ -1.11468 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1.34596 \\ -0.503544 \\ 1.34596 \\ 0 \\ 0 \\ 0 \\ 1.34596 \\ -1.74823 \\ 1.25177 \\ -1.34596 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.759271 \\ 3.54510 \\ 0 \\ 0 \\ -2.12398 \\ -3.33816 \\ -1.11675 \\ -0.454904 \\ 0.454904 \\ 2.12398 \\ -3.54510 \\ 3.33816 \\ -1.21417 \\ 1.78583 \\ -1.21417 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1.00000 \\ 0 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v^{(5)} = \begin{pmatrix} -540624. \\ 21533.1 \\ 288534. \\ 48092.5 \\ 42132.5 \\ 298617. \\ 540624. \\ -272485. \\ -100846. \\ -247489. \\ -21533.1 \\ -148432. \\ -262437. \\ -48092.5 \\ -42132.5 \\ -217340. \\ 184418. \\ 59653.7 \\ 80900.4 \\ 18111.9 \\ 3222.27 \\ 228246. \\ 166213. \\ 122335. \\ 86288.0 \\ -49594.5 \\ -59673.2 \\ -4723.07 \\ -42727.1 \\ -63041.4 \\ -2993.34 \\ -13851.6 \\ -79049.8 \\ -16698.4 \\ -93792.0 \\ -14671.2 \\ 22751.8 \\ 19332.1 \\ 3193.24 \\ 23913.7 \\ 9252.94 \\ 8905.00 \\ -4387.96 \\ -9487.04 \\ -2832.20 \\ -112.807 \\ 470.767 \\ 524.968 \\ 273.650 \\ 62.5934 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 18303.1 \\ -1552.98 \\ -5013.55 \\ -3030.49 \\ -2329.78 \\ -6184.94 \\ -18303.1 \\ 6187.48 \\ -4643.69 \\ 3806.63 \\ 1552.98 \\ -5034.18 \\ 3346.33 \\ 3030.49 \\ 2329.78 \\ -2528.38 \\ -2372.29 \\ 1228.45 \\ 3175.05 \\ -977.775 \\ 269.397 \\ 27.5918 \\ 4906.69 \\ 6701.40 \\ 5893.25 \\ -519.581 \\ -3150.19 \\ -1039.25 \\ -1492.64 \\ -3598.06 \\ -304.620 \\ -2540.28 \\ -1339.23 \\ 617.217 \\ -4042.19 \\ -2418.05 \\ 2587.87 \\ 3191.80 \\ 836.195 \\ 3504.25 \\ 1816.01 \\ 1940.99 \\ -469.469 \\ -1316.06 \\ -684.121 \\ -58.0463 \\ 186.391 \\ 221.749 \\ 128.896 \\ 34.4748 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 1239.34 \\ -310.387 \\ 138.136 \\ -256.758 \\ -613.173 \\ 893.408 \\ -1239.34 \\ -1612.73 \\ 1278.50 \\ -1746.33 \\ 310.387 \\ 971.529 \\ -867.218 \\ 256.758 \\ 613.173 \\ 239.050 \\ 1012.20 \\ 649.459 \\ -580.392 \\ 98.3077 \\ -145.356 \\ 879.644 \\ 613.871 \\ -242.447 \\ -760.650 \\ -1122.99 \\ 240.763 \\ -88.9870 \\ -770.401 \\ 115.493 \\ 117.502 \\ -239.205 \\ -1363.76 \\ -14.9612 \\ 16.1287 \\ -306.455 \\ 667.465 \\ 445.464 \\ 148.466 \\ 668.219 \\ 374.544 \\ 347.886 \\ -217.426 \\ -251.685 \\ -102.127 \\ -25.5771 \\ 84.1410 \\ 97.6492 \\ 57.9091 \\ 17.5913 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 201.162 \\ 0 \\ 0 \\ 0 \\ 606.114 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -744.357 \\ -606.114 \\ -201.162 \\ 0 \\ 0 \\ 0 \\ -201.162 \\ 0 \\ 0 \\ 0 \\ -337.714 \\ 15.5760 \\ 77.9345 \\ 145.648 \\ -239.291 \\ 7.52699 \\ 310.109 \\ 201.162 \\ 228.722 \\ 249.298 \\ 337.714 \\ -15.5760 \\ -77.9345 \\ -145.648 \\ -29.1091 \\ 59.5689 \\ 80.6710 \\ 56.7587 \\ 19.1021 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 187.024 \\ 0 \\ 0 \\ -3.16830 \\ -13.9555 \\ 165.840 \\ -17.1842 \\ -23.8902 \\ -23.8902 \\ 449.740 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -507.189 \\ -517.976 \\ -220.060 \\ -42.4600 \\ 3.16830 \\ 13.9555 \\ -165.840 \\ 8.59208 \\ 21.6347 \\ 2.25548 \\ -257.097 \\ 13.4015 \\ 45.0743 \\ 141.950 \\ -165.840 \\ -48.2426 \\ 212.441 \\ 169.008 \\ 128.548 \\ 183.410 \\ 265.689 \\ -4.80943 \\ -42.8189 \\ -120.315 \\ -26.8030 \\ 50.0521 \\ 69.8774 \\ 51.0521 \\ 17.8253 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 184.407 \\ 242.885 \\ -1158.00 \\ 383.918 \\ 154.557 \\ -849.796 \\ -184.407 \\ -498.924 \\ 93.8644 \\ 327.500 \\ -242.885 \\ 519.804 \\ 989.436 \\ -383.918 \\ -154.557 \\ 649.207 \\ -39.2509 \\ -438.010 \\ 20.6101 \\ 173.141 \\ -27.1460 \\ 449.637 \\ -126.911 \\ -351.239 \\ -139.645 \\ -216.838 \\ -260.136 \\ 63.7681 \\ 125.061 \\ -138.548 \\ 32.9297 \\ 109.743 \\ -24.7858 \\ -158.887 \\ -116.902 \\ 97.2192 \\ 255.963 \\ 100.139 \\ -12.0195 \\ 170.117 \\ 45.8224 \\ 23.8021 \\ -74.7591 \\ -124.900 \\ 4.09762 \\ -5.39706 \\ 43.7952 \\ 45.4569 \\ 22.7657 \\ 6.91552 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 75.9174 \\ 21.0734 \\ 563.551 \\ -523.249 \\ 560.486 \\ -941.082 \\ -75.9174 \\ -255.334 \\ -23.2617 \\ 491.842 \\ -21.0734 \\ -128.332 \\ -476.768 \\ 523.249 \\ -560.486 \\ 796.353 \\ 1006.91 \\ -179.762 \\ -509.384 \\ -268.965 \\ 141.396 \\ 267.102 \\ -347.113 \\ 41.5496 \\ -14.5954 \\ -135.238 \\ -100.358 \\ 5.33038 \\ -38.1775 \\ 310.288 \\ -112.912 \\ 140.226 \\ -350.619 \\ 197.435 \\ -499.238 \\ 182.867 \\ 146.452 \\ 48.7200 \\ -9.57569 \\ 100.636 \\ -32.6006 \\ -39.0608 \\ -81.1160 \\ -25.9703 \\ -24.7710 \\ 6.87601 \\ 30.7561 \\ 25.2702 \\ 7.11547 \\ 0.329027 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} -21.4377 \\ 18.8257 \\ -23.1101 \\ 17.4945 \\ 14.6068 \\ -23.9105 \\ 21.4377 \\ 26.4449 \\ 34.7085 \\ 42.0873 \\ -18.8257 \\ 19.5723 \\ 74.9507 \\ -17.4945 \\ -14.6068 \\ 3.77389 \\ -12.3595 \\ 10.4968 \\ -8.15953 \\ -1.44072 \\ -7.16536 \\ -3.53084 \\ -21.9507 \\ -71.4129 \\ -108.450 \\ -16.5884 \\ 7.45329 \\ -6.21421 \\ -33.9208 \\ 14.4645 \\ 14.0649 \\ 0.929570 \\ -18.3814 \\ -3.28974 \\ 4.81127 \\ -8.67056 \\ 21.8312 \\ -6.58382 \\ 24.3416 \\ 20.5428 \\ 35.4700 \\ 34.0434 \\ -25.0897 \\ -11.6576 \\ -13.6255 \\ -11.9654 \\ 23.0085 \\ 30.1076 \\ 22.9300 \\ 9.68787 \\ 1.00000 \end{pmatrix},$$

$$\begin{pmatrix} -1.32856 \\ 0.468871 \\ -1.54184 \\ -1.39243 \\ -1.28075 \\ -1.39654 \\ 1.32856 \\ 1.63958 \\ 1.24507 \\ -6.54583 \\ -0.468871 \\ -0.543739 \\ 4.11771 \\ 1.39243 \\ 1.28075 \\ 2.12223 \\ 4.79367 \\ 3.67941 \\ 4.06552 \\ 1.92620 \\ 1.43479 \\ -0.583626 \\ 5.82014 \\ -2.03214 \\ 8.58486 \\ 5.53485 \\ -0.197083 \\ -0.198825 \\ 9.63598 \\ 13.6027 \\ 11.9293 \\ 4.27578 \\ -3.20064 \\ -0.709356 \\ -0.541323 \\ -0.264147 \\ -5.20571 \\ -6.92402 \\ -4.16839 \\ 2.39626 \\ -1.22417 \\ 4.32651 \\ -7.99602 \\ -12.4058 \\ -11.1400 \\ -3.93150 \\ 0.503809 \\ -2.01231 \\ 1.49228 \\ -1.73555 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 56.8808 \\ 0 \\ 0 \\ 0 \\ 26.4679 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 120.772 \\ -26.4679 \\ -56.8808 \\ 0 \\ 0 \\ 0 \\ -56.8808 \\ 0 \\ 0 \\ 0 \\ -33.8434 \\ -9.22747 \\ -20.8879 \\ 36.4251 \\ 23.4667 \\ -77.6089 \\ -16.5324 \\ 56.8808 \\ -74.4810 \\ -18.9393 \\ 33.8434 \\ 9.22747 \\ 20.8879 \\ -36.4251 \\ -16.0912 \\ 4.84624 \\ 18.4101 \\ 23.5934 \\ 11.5639 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -3.77181 \\ 0 \\ 0 \\ 1.22755 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3.77181 \\ -1.22755 \\ 6.75586 \\ 3.77181 \\ 0 \\ 0 \\ 6.27533 \\ 9.48354 \\ 9.48354 \\ 3.58230 \\ -1.79115 \\ 0 \\ 0 \\ 0 \\ -3.77181 \\ -5.71174 \\ -3.77181 \\ 1.79115 \\ -0.980659 \\ 3.92059 \\ -5.71174 \\ -9.48354 \\ -9.48354 \\ -3.58230 \\ 0.417063 \\ -1.89759 \\ 1.41706 \\ -1.71238 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 11.7600 \\ 0 \\ 0 \\ 38.7332 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -11.7600 \\ -38.7332 \\ -55.4788 \\ -11.7600 \\ 0 \\ 0 \\ -16.1923 \\ 6.06331 \\ 6.06331 \\ 9.43518 \\ -4.71759 \\ 0 \\ 0 \\ 0 \\ 11.7600 \\ -17.8233 \\ 11.7600 \\ 4.71759 \\ 17.4776 \\ 13.1057 \\ -17.8233 \\ -6.06331 \\ -6.06331 \\ -9.43518 \\ 16.5380 \\ 21.7484 \\ 17.5380 \\ 8.14992 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} -64.8022 \\ -36.4852 \\ 109.148 \\ -40.4431 \\ -8.74395 \\ 81.0525 \\ 64.8022 \\ 48.1453 \\ 12.9528 \\ -9.28428 \\ 36.4852 \\ 9.30007 \\ -87.8168 \\ 40.4431 \\ 8.74395 \\ -49.3031 \\ 8.23444 \\ -33.4049 \\ 14.6346 \\ -22.8788 \\ -8.23963 \\ -25.1250 \\ -22.4651 \\ -30.6315 \\ 3.61285 \\ 59.9803 \\ -75.5421 \\ 32.4215 \\ 58.9247 \\ -51.1407 \\ 4.24215 \\ 39.7834 \\ 60.2089 \\ 8.23497 \\ -49.0909 \\ 47.5135 \\ -16.0487 \\ 39.3805 \\ -14.7206 \\ 6.65676 \\ -11.4945 \\ -2.46136 \\ 19.2566 \\ -38.0807 \\ -0.725033 \\ 0.408211 \\ 13.1834 \\ 17.6943 \\ 9.93487 \\ 3.14403 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} -6.12244 \\ -1.91650 \\ -35.9529 \\ 59.8475 \\ -61.5314 \\ 78.5396 \\ 6.12244 \\ 36.3788 \\ 5.87248 \\ -26.1545 \\ 1.91650 \\ -13.0295 \\ 23.1175 \\ -59.8475 \\ 61.5314 \\ -40.2192 \\ -37.3755 \\ 29.0793 \\ -59.3750 \\ 10.7413 \\ 18.5861 \\ -27.0498 \\ -12.1659 \\ 25.8649 \\ -2.91176 \\ 20.3162 \\ -13.3249 \\ 1.52228 \\ -26.9471 \\ 62.5414 \\ -26.6298 \\ 43.3759 \\ -17.8842 \\ 18.7847 \\ -62.7597 \\ 35.6340 \\ -14.1294 \\ 14.7441 \\ -2.35171 \\ 23.6956 \\ -11.8881 \\ -18.3042 \\ -11.0561 \\ 14.8908 \\ -22.4296 \\ 7.02112 \\ 8.54101 \\ 9.11659 \\ 4.20204 \\ -0.631923 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5.29437 \\ 4.50661 \\ 0 \\ 0 \\ 0 \\ 0 \\ -15.1440 \\ 0 \\ -4.50661 \\ 0 \\ -5.29437 \\ -4.50661 \\ 0 \\ 0 \\ -5.29437 \\ 4.50661 \\ 6.42322 \\ 7.72080 \\ 5.29437 \\ 1.00000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\28.0284\\0\\0\\0\\-50.5118\\0\\0\\0\\0\\0\\-42.0878\\50.5118\\-28.0284\\0\\0\\0\\-28.0284\\0\\0\\0\\10.2960\\6.72993\\10.1389\\17.0470\\51.5311\\52.1628\\13.0128\\28.0284\\-8.50220\\-48.5180\\-10.2960\\-6.72993\\-10.1389\\-17.0470\\-11.3153\\-17.7642\\-7.46194\\9.64051\\8.30223\\1.00000 \end{pmatrix},
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\5.95268\\0\\0\\-4.46193\\10.3948\\-14.2207\\-16.1733\\8.44444\\8.44444\\-25.7731\\0\\0\\0\\0\\0\\-10.7885\\4.06827\\4.22723\\2.55517\\4.46193\\-10.3948\\14.2207\\8.08667\\-7.60725\\-0.837188\\10.4071\\-0.572659\\11.7289\\-5.77623\\14.2207\\-16.1908\\15.0914\\-9.75873\\-5.20357\\-9.33139\\-2.32046\\8.65933\\-12.5661\\-1.83102\\1.14532\\4.53151\\8.58359\\5.53151\\2.05208\\1.00000 \end{pmatrix},
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\-1.58712\\0\\0\\0\\0.517094\\0\\0\\0\\0\\0\\2.49270\\-0.517094\\1.58712\\0\\0\\0\\1.58712\\0\\0\\0\\-0.934530\\1.90214\\3.47594\\4.05768\\2.63793\\-1.21280\\-1.94964\\-1.58712\\0.392025\\1.01683\\0.934530\\-1.90214\\-3.47594\\-4.05768\\-2.22049\\-0.858515\\-0.292778\\0.429002\\-1.43426\\1.00000 \end{pmatrix},
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\2.76401\\0\\0\\2.11175\\0\\0\\0\\0\\0\\0\\0\\0\\0\\-2.76401\\-2.11175\\4.87576\\-2.76401\\0\\0\\0.544860\\0.961141\\-0.961141\\3.52803\\1.76401\\0\\0\\0\\2.76401\\-9.94865\\-2.76401\\-1.76401\\0\\-6.18464\\-4.42062\\-0.961141\\0.961141\\-3.52803\\3.87576\\4.07289\\4.87576\\4.07289\\1.00000 \end{pmatrix},
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\1.69382\\0\\0\\-0.518610\\0\\0\\0\\0\\0\\0\\0\\0\\0\\-1.69382\\0.518610\\1.17521\\-1.69382\\0\\0\\-2.95990\\-2.05365\\2.05365\\1.38764\\0.693822\\0\\0\\0\\1.69382\\-0.602956\\-1.69382\\-0.693822\\0\\2.09087\\2.78469\\2.05365\\-2.05365\\-1.38764\\0.175212\\-1.57226\\1.17521\\-1.57226\\1.00000 \end{pmatrix},
\begin{pmatrix} -18.0256\\-6.40635\\-35.4000\\17.9207\\-18.9761\\-12.3106\\18.0256\\-11.3094\\-18.2433\\27.2583\\6.40635\\17.2731\\18.0824\\-17.9207\\18.9761\\-6.84193\\117.948\\-79.5836\\79.0981\\-45.1223\\-28.1117\\-0.924903\\-8.10579\\0.0445093\\13.2434\\0.213022\\12.2849\\2.14810\\17.2255\\-14.1106\\11.3288\\1.58562\\-52.9692\\26.2701\\-28.4034\\17.9370\\-4.18207\\-4.41013\\1.67652\\16.4689\\-5.41266\\-8.45973\\-26.7122\\32.3434\\-19.4859\\6.92534\\4.92138\\1.17633\\3.11567\\-1.29792\\1.00000 \end{pmatrix},
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\1.53087\\0\\0\\-0.721803\\-1.88397\\-1.65392\\0.815211\\2.94786\\2.94786\\0.933973\\0\\0\\0\\0\\0\\0.864491\\-0.297678\\0.753196\\-0.586521\\0.721803\\1.88397\\1.65392\\-0.407605\\-1.09665\\-1.85121\\-0.962733\\0.812578\\0.236930\\1.29394\\1.65392\\-0.958733\\-1.19356\\-0.932112\\0.481367\\0.880465\\0.555128\\-1.22018\\-2.08814\\-2.39059\\-1.62516\\-0.821084\\-0.137915\\0.178916\\-1.31683\\1.00000 \end{pmatrix},
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\-6.62126\\0\\0\\0\\-5.52462\\0\\0\\0\\0\\0\\-3.05216\\5.52462\\6.62126\\0\\0\\0\\6.62126\\0\\0\\0\\1.01755\\-3.15000\\4.32417\\1.83102\\3.17324\\-8.65954\\6.62445\\-6.62126\\-4.61702\\-5.91874\\-1.01755\\3.15000\\-4.32417\\-1.83102\\1.33383\\1.01224\\4.42392\\3.26812\\1.41168\\1.00000 \end{pmatrix},
$$

$$
\begin{pmatrix} -1.20496\\0.985200\\-4.60608\\-4.86160\\0.889889\\3.86511\\1.20496\\5.30993\\-4.75633\\-4.23127\\-0.985200\\2.17641\\3.12783\\4.86160\\-0.889889\\-1.34620\\10.1984\\1.05807\\-7.19683\\8.83102\\-5.88425\\-1.95348\\1.71237\\-0.698159\\0.481419\\3.53642\\3.42675\\0.347618\\4.33279\\-0.245611\\-1.04845\\1.49111\\-2.41282\\-4.67863\\-2.89093\\-1.41459\\-1.51058\\-0.854136\\-1.50836\\0.183960\\-1.39633\\3.07599\\-3.80089\\-1.06561\\-1.01454\\-0.829502\\1.17490\\-2.45793\\1.45449\\-1.61663\\1.00000 \end{pmatrix},
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\6.00000\\-9.00000\\9.00000\\-18.0000\\0\\0\\0\\0\\0\\-9.00000\\9.00000\\3.00000\\0\\0\\0\\0\\-3.00000\\6.00000\\-3.00000\\12.0000\\3.00000\\-9.00000\\9.00000\\0\\-9.00000\\3.00000\\0\\6.00000\\-2.00000\\-9.00000\\-6.00000\\12.0000\\-15.0000\\6.00000\\2.00000\\3.00000\\3.00000\\-1.00000\\1.00000 \end{pmatrix},
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\3.45641\\-3.22952\\0\\0\\0\\0\\0.834778\\0\\3.22952\\0\\-3.45641\\3.22952\\0\\0\\-3.45641\\-3.22952\\-3.35175\\1.51697\\3.45641\\1.00000 \end{pmatrix},
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix},
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0.338622\\0\\0\\0\\-7.71096\\0\\0\\0\\0\\0\\-4.51513\\7.71096\\-0.338622\\0\\0\\0\\-0.338622\\0\\0\\0\\6.66859\\3.81529\\-8.38428\\12.0253\\-4.42977\\-5.48647\\1.30018\\0.338622\\2.58427\\0.0346812\\-6.66859\\-3.81529\\8.38428\\-12.0253\\5.47214\\0.754448\\1.67740\\2.68529\\-1.07705\\1.00000 \end{pmatrix},
\begin{pmatrix} 2.72171\\7.41010\\3.77090\\-0.564272\\-2.62478\\2.50525\\-2.72171\\3.00248\\5.10102\\2.80371\\-7.41010\\-1.79915\\0.0901724\\0.564272\\2.62478\\-1.94268\\2.77998\\1.40825\\1.55829\\-3.86370\\-4.24259\\-0.143156\\-3.36628\\-2.06192\\3.82272\\-3.05941\\-2.93712\\-2.07683\\0.218775\\0.482331\\1.62156\\0.328899\\-0.125270\\1.48556\\-0.980738\\-0.152113\\2.41224\\-2.79924\\-2.50105\\-0.800490\\-0.135869\\-4.55818\\-3.29641\\-0.803554\\0.613416\\-1.67008\\2.72449\\0.763887\\1.10438\\2.40467\\1.00000 \end{pmatrix},
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\-0.306217\\0\\0\\0\\0.0950432\\0\\0\\0\\0\\0\\-0.250539\\-0.0950432\\0.306217\\0\\0\\0\\0.306217\\0\\0\\0\\-0.118987\\-0.502693\\-0.424112\\0.426805\\0.529829\\0.365784\\0.0466593\\-0.306217\\-0.281928\\1.15753\\0.118987\\0.502693\\0.424112\\-0.426805\\-0.505885\\-0.943674\\-0.125150\\0.188848\\-1.18148\\1.00000 \end{pmatrix},
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\-0.858739\\0\\0\\1.46543\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0.858739\\-1.46543\\0.948409\\0.858739\\0\\0\\-1.49407\\-0.274326\\-0.274326\\2.20884\\-1.10442\\0\\0\\0\\-0.858739\\1.13307\\-0.858739\\1.10442\\1.24568\\-2.23749\\1.13307\\0.274326\\0.274326\\-2.20884\\-0.884676\\-2.55780\\0.115324\\2.45723\\1.00000 \end{pmatrix},
$$

$$
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -3.27379 \\ 0 \\ 0 \\ -3.47575 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3.27379 \\ 3.47575 \\ -2.35450 \\ 3.27379 \\ 0 \\ 0 \\ 0.951203 \\ 1.46845 \\ 1.46845 \\ -1.43839 \\ 0.719196 \\ 0 \\ 0 \\ 0 \\ -3.27379 \\ 1.80535 \\ -3.27379 \\ -0.719196 \\ -2.99299 \\ -1.08615 \\ 1.80535 \\ -1.46845 \\ -1.46845 \\ 1.43839 \\ 0.236439 \\ 2.06926 \\ 1.23644 \\ 0.603396 \\ 1.00000 \end{pmatrix},
\begin{pmatrix} -2.21172 \\ 2.61909 \\ 1.49421 \\ -3.59018 \\ -5.05307 \\ -0.842954 \\ 2.21172 \\ -0.749350 \\ 1.07462 \\ -1.25524 \\ -2.61909 \\ -1.82584 \\ 1.42299 \\ 3.59018 \\ 5.05307 \\ 1.63993 \\ -1.21729 \\ -0.683694 \\ 1.99170 \\ -0.131954 \\ 2.35855 \\ -1.00892 \\ 0.458265 \\ -1.09136 \\ -0.891410 \\ 2.33115 \\ 4.39012 \\ 2.14265 \\ -1.35109 \\ -1.94443 \\ -4.42302 \\ -1.21354 \\ -1.64863 \\ -3.72705 \\ -1.99730 \\ -0.0617016 \\ -1.06763 \\ -0.981073 \\ 0.149882 \\ 1.60446 \\ -0.361147 \\ 0.363536 \\ 0.326438 \\ -0.967027 \\ -0.656706 \\ -0.777664 \\ -0.287768 \\ 0.0538908 \\ -0.592952 \\ -0.911256 \\ 1.00000 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2.42708 \\ 0 \\ 0 \\ 0 \\ 2.90751 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.459169 \\ -2.90751 \\ 2.42708 \\ 0 \\ 0 \\ 0 \\ 2.42708 \\ 0 \\ 0 \\ 0 \\ -2.50745 \\ 2.11828 \\ -1.65228 \\ 0.783303 \\ -1.40429 \\ 2.76040 \\ -0.0272594 \\ -2.42708 \\ -1.61279 \\ -0.155733 \\ 2.50745 \\ -2.11828 \\ 1.65228 \\ -0.783303 \\ 1.00422 \\ -2.86376 \\ -0.307965 \\ 0.592208 \\ 0.555800 \\ 1.00000 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.00000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1.00000 \\ 0 \\ 0 \\ -1.00000 \\ 1.00000 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\begin{pmatrix} -1.25950 \\ -1.70757 \\ -2.15835 \\ 0.486716 \\ 1.56744 \\ -1.62178 \\ 1.25950 \\ 0.887468 \\ 0.0597315 \\ -1.29619 \\ 1.70757 \\ 1.42689 \\ -0.927579 \\ -0.486716 \\ -1.56744 \\ 1.02972 \\ -0.451519 \\ 0.646913 \\ 0.741401 \\ -0.241638 \\ -1.00515 \\ -0.985548 \\ 1.88825 \\ 1.65904 \\ -2.07010 \\ 0.461530 \\ -3.13302 \\ -1.40193 \\ -0.219646 \\ 0.222442 \\ 2.80197 \\ -1.47139 \\ -0.659967 \\ 1.89211 \\ -0.323203 \\ -0.940282 \\ -1.91250 \\ 0.686561 \\ -1.65053 \\ 0.217295 \\ -1.62305 \\ -0.711006 \\ 1.02731 \\ -0.707585 \\ -1.00045 \\ 1.35635 \\ -0.235986 \\ 1.14109 \\ 0.451078 \\ 0.118851 \\ 1.00000 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1.51220 \\ 0.794465 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.260417 \\ 0 \\ -0.794465 \\ 0 \\ 1.51220 \\ -0.794465 \\ 0 \\ 0 \\ 1.51220 \\ 0.794465 \\ -2.39516 \\ 1.65558 \\ -1.51220 \\ 1.00000 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
$$

$$
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1.06150 \\ 0 \\ 0 \\ 3.24978 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.06150 \\ -3.24978 \\ 2.18828 \\ 1.06150 \\ 0 \\ 0 \\ 2.73486 \\ -5.63791 \\ 5.63791 \\ -4.12300 \\ -2.06150 \\ 0 \\ 0 \\ 0 \\ -1.06150 \\ -1.80014 \\ 1.06150 \\ 2.06150 \\ 0 \\ -1.86164 \\ -3.92314 \\ 5.63791 \\ -5.63791 \\ 4.12300 \\ 1.18828 \\ -1.38814 \\ 2.18828 \\ -1.38814 \\ 1.00000 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.23108 \\ 1.72883 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2.13796 \\ 0 \\ -1.72883 \\ 0 \\ -1.23108 \\ -1.72883 \\ 0 \\ 0 \\ -1.23108 \\ 1.72883 \\ -1.66466 \\ -1.47330 \\ 1.23108 \\ 1.00000 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.144346 \\ 0 \\ 0 \\ 0.0495503 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.144346 \\ -0.0495503 \\ 0.129039 \\ -0.144346 \\ 0 \\ 0 \\ -0.540138 \\ 0.259027 \\ 0.259027 \\ -1.78792 \\ 0.893960 \\ 0 \\ 0 \\ 0 \\ 0.144346 \\ -0.403373 \\ 0.144346 \\ -0.893960 \\ 0.250385 \\ 1.29733 \\ -0.403373 \\ -0.259027 \\ -0.259027 \\ 1.78792 \\ 0.693125 \\ -2.36230 \\ 1.69313 \\ -1.49817 \\ 1.00000 \end{pmatrix},
\begin{pmatrix} -0.196390 \\ -0.161869 \\ 0.00479129 \\ 1.14660 \\ -0.905467 \\ -0.457135 \\ 0.196390 \\ -0.380500 \\ 0.772006 \\ -0.270147 \\ 0.161869 \\ -0.472150 \\ 3.41435 \\ -1.14660 \\ 0.905467 \\ -0.201760 \\ 1.71760 \\ -0.108743 \\ -1.33253 \\ -1.49662 \\ 1.09830 \\ -0.218426 \\ 0.929041 \\ -2.94699 \\ 1.76661 \\ 0.563021 \\ -0.684881 \\ -0.0627895 \\ 2.54049 \\ -5.04548 \\ 5.15721 \\ -4.24637 \\ -2.06089 \\ 0.644557 \\ -0.249447 \\ 0.0120105 \\ -0.925273 \\ -1.59126 \\ 0.984736 \\ 1.62652 \\ 0.0160683 \\ -1.36226 \\ -3.61975 \\ 4.69610 \\ -4.64965 \\ 3.83235 \\ 1.14571 \\ -1.58307 \\ 2.11488 \\ -1.40582 \\ 1.00000 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.531657 \\ 0 \\ 0 \\ 0 \\ 1.64553 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.53813 \\ -1.64553 \\ -0.531657 \\ 0 \\ 0 \\ 0 \\ -0.531657 \\ 0 \\ 0 \\ 0 \\ 1.13603 \\ -2.26145 \\ 2.47509 \\ 3.75603 \\ -6.21338 \\ 3.15174 \\ -0.583900 \\ 0.531657 \\ -4.20349 \\ 4.02444 \\ -1.13603 \\ 2.26145 \\ -2.47509 \\ -3.75603 \\ 3.43182 \\ -1.75175 \\ -0.994631 \\ 1.84390 \\ -1.24288 \\ 1.00000 \end{pmatrix},
\begin{pmatrix} 7.18176 \\ -13.3191 \\ -5.67620 \\ 14.0652 \\ 17.4927 \\ 0.000327867 \\ -7.18176 \\ 1.22428 \\ -1.80230 \\ 6.46761 \\ 13.3191 \\ 8.74614 \\ -4.57951 \\ -14.0652 \\ -17.4927 \\ -4.45391 \\ 10.6312 \\ 5.58337 \\ 5.72223 \\ 13.0568 \\ 10.6123 \\ -2.19436 \\ -2.01403 \\ 1.50957 \\ -3.56591 \\ 1.87238 \\ 7.36969 \\ 3.58449 \\ 2.82345 \\ -2.59928 \\ -6.60514 \\ 0.294856 \\ 1.33204 \\ -7.34471 \\ -5.32537 \\ -3.97925 \\ 1.22706 \\ 3.06243 \\ 1.44472 \\ -3.55788 \\ -2.66883 \\ -0.576593 \\ -4.31066 \\ -0.580834 \\ -0.764641 \\ -1.41808 \\ 3.47019 \\ -1.09355 \\ -2.54354 \\ 0.339663 \\ 1.00000 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.603661 \\ 0 \\ 0 \\ -0.842916 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.603661 \\ 0.842916 \\ -0.239254 \\ -0.603661 \\ 0 \\ 0 \\ 0.680180 \\ -0.269581 \\ 0.269581 \\ -0.792677 \\ -0.396339 \\ 0 \\ 0 \\ 0 \\ 0.603661 \\ -0.648248 \\ -0.603661 \\ 0.396339 \\ 0 \\ 0.955413 \\ 0.559074 \\ 0.269581 \\ -0.269581 \\ 0.792677 \\ -1.23925 \\ -0.112497 \\ -0.239254 \\ -0.112497 \\ 1.00000 \end{pmatrix},
$$

$$
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\-0.469660\\-0.800394\\0\\0\\0\\0\\-0.568300\\0\\0.800394\\0\\0.469660\\0.800394\\0\\0\\0.469660\\-0.800394\\-0.0116497\\-0.420050\\-0.469660\\1.00000 \end{pmatrix},
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix},
\begin{pmatrix} 0\\0\\0\\0\\0\\0\\3.49231\\0\\0\\2.35203\\5.44463\\2.03460\\2.54229\\-2.50211\\-2.50211\\1.09959\\0\\0\\0\\0\\0\\-1.88740\\1.20519\\0.0799250\\-1.50865\\-2.35203\\-5.44463\\-2.03460\\-1.27114\\-0.930815\\3.43292\\-0.347836\\-0.641424\\3.95982\\-0.467506\\-2.03460\\-1.60779\\0.660636\\-0.317432\\0.173918\\0.0411444\\-0.923308\\-0.629720\\-0.526898\\-0.463309\\1.28285\\-0.762493\\0.676972\\0.237507\\-0.560535\\1.00000 \end{pmatrix}.
$$

References

[1] P. Pearce, J. Rasmussen, and I. Tipunin. *Critical dense polymers with Robin boundary conditions, half-integer Kac labels and $\mathbb{Z}_4$ fermions*. Nuclear Physics, Section B **889**(C), 580 (2014). URL `http://www.sciencedirect.com/science/article/pii/S0550321314003289`.

[2] A. Morin-Duchesne, P. Pearce, and J. Rasmussen. *Fusion hierarchies, T-systems, and Y-systems of logarithmic minimal models*. Journal of Statistical Mechanics: Theory and Experiment **2014**(5) (2014). URL `http://stacks.iop.org/1742-5468/2014/i=5/a=P05012`.

[3] C. Yang. *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*. Physical Review Letters **19**, 1312 (1967). URL `https://link.aps.org/doi/10.1103/PhysRevLett.19.1312`.

[4] R. Baxter. *Partition function of the eight-vertex lattice model*. Annals of Physics **70**(1), 193 (1972). URL `http://www.sciencedirect.com/science/article/pii/0003491672903351`.

[5] F. Calogero. *Integrable systems: Overview*. In J.-P. Franoise, G. L. Naber, and T. S. Tsun, eds., *Encyclopedia of Mathematical Physics*, pp. 106 – 123 (Academic Press, Oxford, 2006). URL `http://www.sciencedirect.com/science/article/pii/B012512666200170X`.

[6] L. Samaj and Z. Bajnok. *Introduction to the Statistical Physics of Integrable Many-Body Systems* (Cambridge University Press, Cambridge, 2013).

[7] L. Onsager. *Crystal statistics. I. a two-dimensional model with an order-disorder transition*. Physical Review **65**, 117 (1944). URL `https://link.aps.org/doi/10.1103/PhysRev.65.117`.

[8] E. Ising. *Beitrag zur Theorie des Ferromagnetismus*. Zeitschrift fur Physik **31**, 253 (1925).

[9] E. Lieb. *Residual entropy of square ice*. Physical Review **162**, 162 (1967). URL `https://link.aps.org/doi/10.1103/PhysRev.162.162`.

[10] B. Sutherland, C. Yang, and C. Yang. *Exact solution of a model of two-dimensional ferroelectrics in an arbitrary external electric field*. Physical Review Letters **19**, 588 (1967). URL `https://link.aps.org/doi/10.1103/PhysRevLett.19.588`.

[11] B. Nienhuis, C. Domb, and J. Lebowitz. *Coulomb gas formulation of two-dimensional phase transitions*. Phase Transitions and Critical Phenomena **11**, 1 (1987).

[12] H. Blote and B. Nienhuis. *Critical behaviour and conformal anomaly of the $O(n)$ model on the square lattice*. Journal of Physics A: Mathematical and General **22**(9), 1415 (1989). URL `http://stacks.iop.org/0305-4470/22/i=9/a=028`.

[13] V. Jones. *Planar algebras I* (1999). URL `https://arxiv.org/abs/math/9909027`.

[14] H. Temperley and E. Lieb. *Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: Some exact results for the 'percolation' problem*. Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences (1934-1990) **322**(1549), 251 (1971).

[15] P. Pearce, J. Rasmussen, and J.-B. Zuber. *Logarithmic minimal models*. Journal of Statistical Mechanics: Theory and Experiment **2006**(11), P11017 (2006). URL `http://stacks.iop.org/1742-5468/2006/i=11/a=P11017`.

[16] P. Pearce and J. Rasmussen. *Solvable critical dense polymers*. Journal of Statistical Mechanics: Theory and Experiment **2007**(02), P02015 (2007). URL `http://stacks.iop.org/1742-5468/2007/i=02/a=P02015`.

[17] E. Sklyanin. *Boundary conditions for integrable quantum systems*. Journal of Physics A: Mathematical and General **21**(10), 2375 (1988).

[18] J. Cardy. *Boundary conditions, fusion rules and the Verlinde formula*. Nuclear Physics, Section B **324**(3), 581 (1989). URL `http://www.sciencedirect.com/science/article/pii/055032138990521X`.

[19] R. Behrend and P. Pearce. *Integrable and conformal boundary conditions for $s\ell(2)$ A-DE lattice models and unitary minimal conformal field theories*. Journal of Statistical Physics **102**(3), 577 (2001). URL `https://doi.org/10.1023/A:1004890600991`.

[20] M. Wadati, T. Deguchi, and Y. Akutsu. *Exactly solvable models and knot theory*. Physics Reports **180**(4), 247 (1989). URL `http://www.sciencedirect.com/science/article/pii/0370157389901233`.

[21] H. Saleur. *Geometrical lattice models for $N = 2$ supersymmetric theories in two dimensions*. Nuclear Physics B **382**(3), 532 (1992). URL `http://www.sciencedirect.com/science/article/pii/055032139290658X`.

[22] P. Martin and H. Saleur. *The blob algebra and the periodic Temperley-Lieb algebra*. Letters in Mathematical Physics **30**(3), 189 (1994). URL `https://doi.org/10.1007/BF00805852`.

[23] L. Kauffman and S. Lins. *Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds* (Princeton University Press, 1994). URL `http://www.jstor.org/stable/j.ctt1bgzb7v`.

[24] V. Jones. *Index for subfactors.* Inventiones mathematicae **72**(1), 1 (1983).

[25] H. Wenzl. *Hecke algebras of type A_n and subfactors.* Inventiones mathematicae **92**(2), 349 (1988).

[26] P. Fendley. *Exact S-matrices for supersymmetric sigma models and the Potts model.* Journal of Physics A: Mathematical and General **35**(50), 10675 (2002). URL `http://stacks.iop.org/0305-4470/35/i=50/a=301`.

[27] A. Ong. *Boundary conditions for logarithmic superconformal models.* Master's thesis, University of Melbourne (2012). URL `https://pdfs.semanticscholar.org/c5bb/82049355e411a4a878d8d109aebb2e64758a.pdf`.

[28] P. Pearce, J. Rasmussen, and E. Tartaglia. *Logarithmic superconformal minimal models.* Journal of Statistical Mechanics: Theory and Experiment **2014**(5), P05001 (2014). URL `http://stacks.iop.org/1742-5468/2014/i=5/a=P05001`.

[29] D. Ridout and Y. Saint-Aubin. *Standard modules, induction and the structure of the Temperley-Lieb algebra.* Advances In Theoretical And Mathematical Physics **18**(5), 957 (2014). URL `https://projecteuclid-org.ezproxy.library.uq.edu.au/download/pdf_1/euclid.atmp/1416929529`.

[30] P. Martin. *Potts Models and Related Problems in Statistical Mechanics* (WORLD SCIENTIFIC, 1991). `https://www.worldscientific.com/doi/pdf/10.1142/0983`, URL `https://www.worldscientific.com/doi/abs/10.1142/0983`.

[31] S. Weinberg. *The Quantum Theory of Fields*, vol. 1 (Cambridge University Press, 1995).

[32] M. Rayes, V. Trevisan, and P. Wang. *Factorization properties of Chebyshev polynomials.* Computers and Mathematics with Applications **50**(8), 1231 (2005). URL `http://www.sciencedirect.com/science/article/pii/S0898122105003767`.

[33] A. Morin-Duchesne, J. Rasmussen, P. Ruelle, and Y. Saint-Aubin. *On the reality of spectra of $U_q(s\ell_2)$-invariant XXZ Hamiltonians.* Journal of Statistical Mechanics: Theory and Experiment **2016**(5), 053105 (2016). URL `http://stacks.iop.org/1742-5468/2016/i=5/a=053105`.